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English for solving mathematical problems

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Предлагаемое пособие является дополнением к основным учебным пособиям курса английского языка для механиков и математиков. Его цель – помочь студентам научиться использовать английский язык в ходе решения задач по специальности. Это предполагает овладение практическими навыками анализа предложенного задания, выбора стратегии его выполнения, оценки правильности выполнения задания и, наконец, умение объяснить ход решения.

В пособии представлены образцы математических текстов, включающих постановку задачи и ход ее решения. Лексика анализируется и закрепляется с помощью упражнений, заключительные из которых предлагают применить приобретенные навыки и умения для решения конкретных задач.

В качестве текстовой основы использованы отрывки из книги Jordan D.W., Smith P., Mathematical Techniques. Oxford: OUP, 2008.

Unit 1

1. Read the text and make a note of any useful word combination you find there.

14.1 **Reversing differentiation**
Compare the following two problems:
Problem A: $\frac{d}{dx}\sin x = f(x)$
Problem B: $\frac{d}{dx}F(x) = \cos x$
For Problem A we know already that

$$f(x) = \cos x.$$

This provides one answer to Problem B, which is solved by

$$F(x) = \sin x.$$

Since $\cos x$ is the derivative of $\sin x$, we say that $\sin x$ is an antiderivative of $\cos x$ (we say an antiderivative because it is not the only one; for example, $\sin x + 1$ is also an antiderivative).

The antidifferentiation question in Problem B can be expressed in various ways; for example,

- What must be differentiated to get $\cos x$?
- What curves have slope equal to $\cos x$ at every point?
- Find y as a function of x if $\frac{dy}{dx} = \cos x$.

Finding antiderivatives is the opposite or inverse process to that of finding derivatives.

The following examples show that a function $f(x)$ has an infinite number of antiderivatives: there is an infinite number of functions whose derivatives are $f(x)$. However, they are all very simple variants on a single function.

- Example 14.1: *Find y as a function of x if* $\frac{dy}{dx} = 2x$

  One solution is $y = x^2$, because its derivative is $2x$. But the derivatives of x^2+3, $x^2-1/2$, and so on are also equal to $2x$. In fact $y = x^2+C$ is an antiderivative of $2x$ for any constant C.

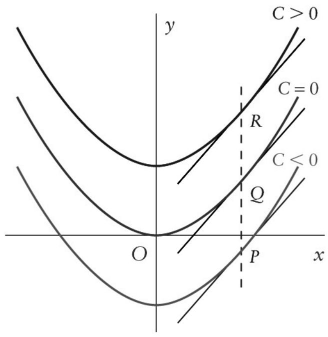

Fig. 14.1

Some of these solutions are shown in Fig. 14.1. Different choices for C just shift the graph bodily up or down parallel to itself. Therefore, at any particular value of x, such as is represented by the vertical line PQR, the slopes are all the same, independently of the value of C.

- Example 14.2: *Find a collection of antiderivatives of* $\sin 2x$.

  We want y such that $\frac{dy}{dx} = \sin 2x$. If we differentiate a cosine we get something involving a sine, so first of all test whether $y = \cos 2x$ is close to being an antiderivative of $\sin 2x$. We find that $\frac{dy}{dx} = -2 \sin 2x$. This contains an unwanted factor (-2). It can be eliminated by choosing instead

$$y = \frac{1}{-2} \cos 2x = -\frac{1}{2} \cos 2x,$$

  for then we have $\frac{dy}{dx} = -\frac{1}{2}(-2 \sin 2x) = \sin 2x$, which is right. Therefore, one antiderivative is $-1/2 \cos 2x$, and the rest are of the form $y = -\frac{1}{2} \cos 2x + C$ (C is any constant).

2. Find in the text you have just read a word which:

- means "*used for referring to something that you are going to say or mention next, especially a list of people or things*".
- means "*used when explaining why someone does something or why a situation exists*".
- means "*used for talking about reasons or causes*".

3. Complete the text with the words and phrases in the box.

from this, that is to say, then, to avoid, solve, try, find

- Example 14.3: ... *the equation* $\frac{dy}{dx} = e^{-3x}$ *(..., ... a collection of antiderivatives of* e^{-3x}*).*

  ... $y = e^{-3x}$; ... $\frac{dy}{dx} = -3e^{-3x}$. ... the unwanted factor (-3) we should have taken

$$y = \frac{1}{-3} e^{-3x} = -\frac{1}{3} e^{-3x}.$$

  ... we construct an infinite collection of antiderivatives: $-\frac{1}{3} e^{-3x} + C$ (C is any constant).

4. In the text that follows, find word combinations with the noun *process*.

It can be proved that the above process, of finding a particular antiderivative of a function and adding constants, generates all possible antiderivatives for that function.

Antiderivatives of $f(x)$

A function $F(x)$ is called an antiderivative of $f(x)$ if

$$\frac{d}{dx}F(x) = f(x).$$

If $F(x)$ is any particular antiderivative of $f(x)$, then all the antiderivatives are given by

$$F(x) + C, \tag{14.1}$$

where C can be any constant. (Therefore, any two antiderivatives differ by a constant).

An antiderivative of a function is also more usually called indefinite integral, and the process of getting it is called integration. If you know the term already, it is perfectly safe to use it.

5. The following two examples show the importance in practice of including the constant C. Study them.

- Example 14.6: *A point is at $x = 2$ on the x axis at time $t = 0$, then moves with velocity $v = t - t^2$. Find where it is at time $t = 3$.*

  Velocity is the rate at which displacement x changes with time: $v = \frac{dx}{dt}$.

  In this case

  $$v = \frac{dx}{dt} = t - t^2.$$

  Therefore x is some antiderivative of $t - t^2$. All of its antiderivatives are included in

  $$x = \frac{1}{2}t^2 - \frac{1}{3}t^3 + C,$$

  where C is any constant.

  To find what value C must take in this case, we obviously have to take the starting point into consideration: $x = 2$ when $t = 0$. To obtain the value of C, substitute these values into our expression:

  $$2 = 0 - 0 + C.$$

  Therefore $C = 2$, so the position at any time is given by

  $$x = \frac{1}{2}t^2 - \frac{1}{3}t^3 + 2.$$

  Finally, when $t = 3$, we have $x = -\frac{5}{2}$.

- Example 14.7: *Find the equation of the curve which passes through the point* $(\pi, -1)$ *and whose slope is given by* $\frac{dy}{dx} = \sin 2x$.

  Since the required y is an antiderivative of $\sin 2x$, the equation of the curve must take the form

  $$y = -\frac{1}{2}\cos 2x + C,$$

  where C is some (not 'any') constant. Since also we know that the curve passes through the point $x = \pi$, $y = -1$, we must require

  $$-1 = -\frac{1}{2}\cos 2\pi + C = -\frac{1}{2} + C,$$

  so $C = -\frac{1}{2}$. Finally the required curve is

  $$y = -\frac{1}{2}\cos 2x - \frac{1}{2}.$$

6. Use the following word combinations in the sentences of your own.

- to take into account
- the process of getting smth.
- it can be proved that
- to provide an answer to ...
- to be equal to
- that is to say
- from this we construct

7. Read the text. Pay special attention to the conjunction *given that* meaning *considering the stated facts*, think of its Russian equivalent. Use it in the sentences of your own.

- Example 14.8: *Obtain the antiderivative of* $(3x-2)^3$.

  As in the earlier examples, we try to guess the structure of y, given that $\frac{dx}{dy} = (3x-2)^3$. There is not much to go on, so try an analogy with x^3; it would lead us to try something like $y = (3x-2)^4$. To check this, differentiate using the chain rule with $u = 3x - 2$ and $y = u^4$:

  $$\frac{dy}{dx} = 4(3x-2)^3 \cdot 3 = 12(3x-2)^3.$$

  The factor 12 is unwanted; we really needed $y = \frac{1}{12}(3x-2)^4$. Therefore all the antiderivatives are given by $y = \frac{1}{12}(3x-2)^4 + C$.

8. Read the text. Use a dictionary if necessary. Then do the exercise that follows.

Signed area generated by a graph

Figure 14.2 shows the graph of a function $y = f(x)$ between $x = a$ and $x = b$, in which we assume that the x and y scales are the same. Divide the range as shown into N sections so that in any section y is either positive only, or negative only.

Let A_1, A_2, ... denote the geometrical areas of these segments, and A the sum of these. Geometrical area is always positive, so A_1, A_2, ... are all positive numbers. Then

$$A = A_1 + A_2 + A_3 + ... + A_N \tag{14.3}$$

is naturally called 'the geometrical area between the curve and the x axis'.

We require a different quantity, $\mathcal{A}$, called the signed area between the curve and the x axis. This is defined by

$$\mathcal{A} = A_2 - A_2 + A_3 - \ldots - A_N. \tag{14.4}$$

In forming $\mathcal{A}$, we use the rule: if y is positive, the contribution takes a positive sign; if y is negative, the contribution takes a negative sign. This quantity has a far more useful range of applications than has geometrical area. For example, suppose that a point is moving on a straight line; then the signed displacement from its starting point is equal to the signed area of its velocity-time graph.

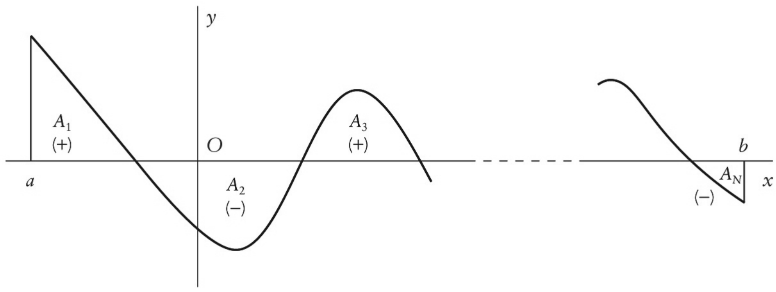

Fig. 14.2

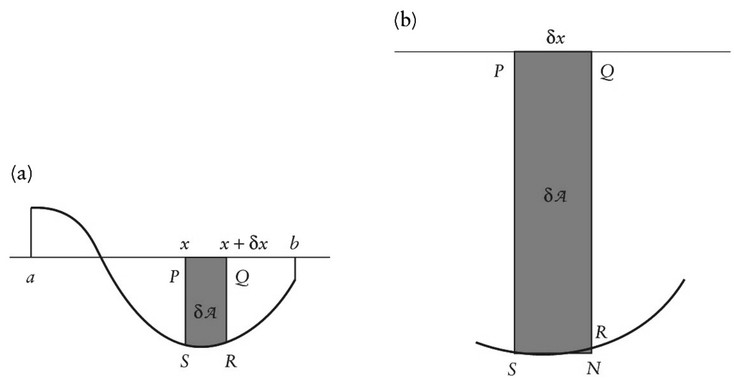

Fig. 14.3

We show how to calculate the signed area $\mathcal{A}$ of the graph of $y = f(x)$ between two given points, $x = a$ and $x = b$. Let $\mathcal{A}(x)$ represent the signed area between a and a variable point with coordinate x (Fig 14.3a). Increase x by a small step δx; the signed area from a to $x + \delta x$ is $\mathcal{A}(x+\delta x)$. The change in signed area, $\delta\mathcal{A} = \mathcal{A}(x+\delta x) - \mathcal{A}(x)$ (positive or negative), is equal to the signed area of $PQRS$ in Figs 14.3a and b. This is very nearly equal to the signed area of the rectangle $PQNS$ in Fig . 14.3b (in this case the required sign is negative) so

$$\delta\mathcal{A} \approx f(x)\delta x$$

which automatically takes the right sign. Therefore

$$\frac{\delta\mathcal{A}}{\delta x} \approx f(x).$$

Now let $\delta x \to 0$; '$\approx$' becomes '$=$', and $\frac{\delta\mathcal{A}}{\delta x}$ becomes $\frac{d\mathcal{A}}{dx}$, so that

$$\frac{d\mathcal{A}}{dx} = f(x). \tag{14.5}$$

From (14.5) $\mathcal{A}(x)$ must be one of the antiderivatives of $f(x)$. To find which one, choose any particular antiderivative and call it $F(x)$. Then $\mathcal{A}(x)$ can differ from $F(x)$ only by a constant, k say, so that

$$\mathcal{A}(x) = F(x) + k. \tag{14.6}$$

To determine the value of k, use the fact that $\mathcal{A}(x) = 0$ at $x = a$, because the starting point is then the same as the end-point; that is to say,

$$\mathcal{A}(a) = 0.$$

Therefore, from (14.6)

$$\mathcal{A}(a) = 0 = F(a) + k,$$

or

$$k = -F(a), \tag{14.7}$$

a known quantity, since we selected the antiderivative $F(x)$ of $f(x)$ ourselves. The required area $\mathcal{A}$ between a and b is given by

$$\mathcal{A} = \mathcal{A}(b) = F(b) - F(a),$$

by putting $x = b$ into (14.6), with (14.7) as the value of k.

The signed area $\mathcal{A}$ of $f(x)$ between area $x = a$ and b

$$\mathcal{A} = F(b) - F(a), \tag{14.8}$$

where $F(x)$ is any antiderivative of $f(x)$.

In practice we naturally use the simplest antiderivative, in which the C in the table is zero. But any nonzero choice of C will cancel out and disappear, since it will be present in both $F(a)$ and $F(b)$.

- Example 14.11 *Find the signed area of $y = x^2$ from $x = -1$ to $x = 2$.* (This happens to be the same as the geometrical area, because y is never negative.) Here $a = -1$ and $b = 2$. Also, the simplest antiderivative of x^2 is

$$F(x) = \frac{1}{3}x^3.$$

Therefore, from (14.8),

$$\mathcal{A} = F(b) - F(a) = \frac{1}{3}(2)^3 - \frac{1}{3}(-1)^3 = 3.$$

There is a special notation, **the square-bracket notation**, which we shall use generally from now onward.

Square-bracket notation

$$[F(x)]_a^b \text{ stands for } F(b) - F(a). \tag{14.9}$$

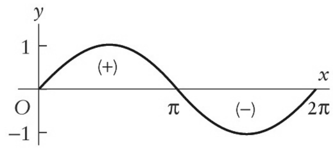

Fig. 14.4

- Example 14.12 *Find (a) signed area, and (b) the geometrical area, between* $y = \sin x$ *and the* x *axis from* $x = 0$ *to* $x = 2\pi$*.*

  (a) $f(x) = \sin x$, so $F(x) = -\cos x$ is an antiderivative. From (14.8) and (14.9), with $a = 0$ and $b = 2\pi$, the signed area A is given by

  $$A = [-\cos x]_0^{2\pi} = -[\cos x]_0^{2\pi} = -(\cos 2\pi - \cos 0) = 0,$$

  as is expected from Fig. 14.4: the positive and negative sections cancel.

  (b) The geometrical area A can be obtained by splitting the range into a positive section 0 to π, and a negative section from π to 2π (see Fig. 14.4). The negatively signed section π to 2π must have its sign reversed in order to give the geometrical area:

  $$A = [\text{geometrical area of 1st loop}] + [\text{geometrical area of 2nd loop}] =$$

  $$= [\text{signed area of 1st loop}] - [\text{signed area of 2nd loop}].$$

  This is equal to

  $$[F(x)]_0^{\pi} - [F(x)]_{\pi}^{2\pi} = [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} =$$

  $$= (-\cos\pi + \cos 0) - (-\cos 2\pi + \cos\pi) = (1+1) - (-1 + (-1)) = 2 + 2 = 4.$$

9. Say it in English.

1. Однако полученный результат совсем не тот, который нам был нужен.
2. Разделите обе величины на константу a. Не забудьте ввести произвольную константу c. Пусть начальные условия заданы.
3. Для того чтобы проверить правильность формулы, все, что нам нужно сделать, так это продифференцировать полученный результат.

4. На практике в таком случае мы используем гораздо более простой способ решения. Он заключается в следующем.

5. Оказывается, что это уравнение такое же, как и рассмотренное выше.

6. Найдите функцию, которая удовлетворяет следующему уравнению.

7. Теперь предположим, что x имеет отрицательное значение.

8. Буква c означает "любая константа" или "произвольная константа".

10. Solve the problems below. Use phrases from the texts you have read to help you. Describe your solutions.

14.1 Obtain all antiderivatives of the following functions, and check their correctness by differentiating your results.

(a) x^5; $3x^4$; $2x^3$; $\frac{1}{3}x^2$; $6x$; $f(x) = 3$; $f(x) = 0$.

(b) $-\frac{1}{2}x^{-3}$; $2x^{-2}$; $3x^{-1}$ when $x > 0$ (if in boubt, see (14.2)).

(c) $x^{\frac{3}{2}}$; $x^{\frac{1}{2}}$; $x^{-\frac{1}{2}}$; $x^{\frac{4}{3}}$; $x^{-\frac{1}{3}}$.

(d) $\frac{1}{x^2}$ write as x^{-2}; $\frac{1}{x^4}$; $\frac{1}{x}$ when $x < 0$ (see (14.2)).

(e) $\sqrt{x}(= x^{\frac{1}{2}})$; $\frac{1}{\sqrt{x}}$; $\frac{1}{x^{\frac{1}{2}}}$.

(f) $3x$; $\frac{1}{2}x^2$; $\frac{1}{3x^2}$; $\frac{3}{4x^{\frac{1}{4}}}$.

(g) e^x; e^{-x}; $5e^{2x}$; $e^{-\frac{1}{2}x}$; $3e^{-2x}$.

(h) $\cos x$; $\cos 3x$; $\sin x$; $\sin 3x$.

(i) $1 - 3x$; $1 + 2x - 3x^2$; $3x^4 - 4x^2 + 5$.

(j) $x(x+1)$ (expand by removing the brackets); $(1+2x)(1-2x)$; $(x+1)^2$; $(1+x)(1-\frac{1}{x})$; $x^2(x + x^2)$.

(k) $\frac{x+1}{x}$ (turn it into the sum of two terms); $\frac{2\sqrt{x}-1}{\sqrt{x}}$ (put $\sqrt{x} = x^{\frac{1}{2}}$ and $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, then simplify as the sum of two terms); $\frac{(x+1)^2}{x^3}$.

(l) $e^x + e^{-x}$; $2e^{2x} - 3e^{3x}$; $e^{\frac{1}{2}x}(1 + e^{-\frac{1}{2}x})$; $\frac{1}{e^{2x}}(= e^{-2x})$; $\frac{e^{2x}+e^{-2x}}{e^{2x}}$.

(m) $2\cos 2x$; $3\sin\frac{1}{2}x - 4\cos\frac{1}{3}x$; $2 + \sin 2x$.

14.2 Find all antiderivatives of the following by trial and error, as explained in the text. Confirm your answers by differentiation.

(a) $(x + 1)^3$ (start by trying $(x + 1)^4$); $(3x + 1)^3$; $(3x - 8)^3$.

(b) $(1 - x)^4$; $(8 - 3x)^{\frac{1}{2}}$; $(1 - x)^{\frac{1}{3}}$;

(c) $(2x + 1)^{-2}$; $(1 - x)^{-\frac{1}{2}}$; $\frac{2}{(3x+1)^3}$; $\frac{1}{4(1-x)^{\frac{1}{4}}}$.

(d) $\cos(3x - 2)$ (try first $\sin(3x - 2)$); $3\sin(1 - x)$; $2\sin(2 - 3x)$.

14.3 Find the antiderivatives of the following.

(a) $\frac{1}{x+1}$; $\frac{1}{x-1}$; $\frac{3}{3x-2}$; $\frac{2}{5x-4}$.

(b) $\frac{1}{1-x}$; $\frac{1}{4-5x}$.

(c) $\frac{x}{x+1}$ (it can be written as $1-\frac{1}{x+1}$).

(d) $\frac{x+1}{x-1}$ (compare (c)).

14.4 Use the identities $\cos^2 A = \frac{1}{2}(1+\cos 2A)$, $\sin^2 A = \frac{1}{2}(1-\cos 2A)$, and $\sin A \cos A = \frac{1}{2}\sin 2A$ to get rid of the squares and products in the following expressions, and in that way obtain the antiderivatives.

(a) $\cos^2 x$; $\sin^2 x$; $\sin x \cos x$.

(b) $3\cos^2 2x$; $\sin^2 3x$; $\sin 2x \cos 2x$.

(c) $\cos^4 x$ (you will have to use the identities twice).

14.5 (a) Show that $\frac{d}{dx}(xe^x) = e^x + xe^x$. By rearranging the terms, show that the antiderivatives of xe^x are $e^x(x-1)+C$ (use the fact that e^x can be written as $\frac{d}{dx}e^x$). Confirm the result by differentiation.

(b) Differentiate x^2e^x. By rearranging the terms and using the results in (a), find the antiderivatives of x^2e^x.

14.6 Use the result (14.8) to obtain the signed area between the given graphs and the x axis. By roughly sketching the graphs of the functions for which you obtain zero, explain this fact.

(a) $y = x$, $0 \le x \le 2$;

(b) $y = x$, $-1 \le x \le 1$;

(c) $y = -x^2$, $0 \le x \le 1$;

(d) $y = \cos x$, $-\pi \le x \le \pi$;

(e) $y = \cos x - 1$, $0 \le x \le 2\pi$;

(f) $y = x^{-1}$, $-2 \le x \le -1$;

(g) $y = \sin 3x$, $0 \le x \le \frac{2}{3}\pi$;

(h) $y = \frac{1}{1-x}$, $2 \le x \le 3$ (note: $1-x$ is negative over this range, so make sure you understand Example 14.10; alternatively, write $\frac{1}{1-x} = -\frac{1}{x-1}$).

14.7 Obtain the geometric area between the graph and the x axis in each of the following cases. It is necessary to treat each positive and negative section separately.

(a) $y = -3$, $0 \le x \le 1$ (this is negative all the way);

(b) $y = x^3$, $-1 \le x \le 1$;

(c) $y = 4 - x^2$, $-1 \le x \le 3$;

(d) $y = \cos x$, $0 \le x \le 2\pi$.

14.8 Find the most general function which satisfies the following equations.
(Note: $\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt}$, $\frac{d^3x}{dt^3} = \frac{d}{dt}\frac{d^2x}{dt^2}$, etc. Work in several steps, finding the next lowest derivative in each step.)

(a) $\frac{d^2x}{dt^2} = 0$;

(b) $\frac{d^2x}{dt^2} = t$;

(c) $\frac{d^2t}{dt^2} = \sin t$;

(d) $\frac{d^3x}{dt^3} = 0$;

(e) $\frac{d^2t}{dt^2} = \cos t$;

(f) $\frac{d^3x}{dt^3} = g$ (g is a constant);

(g) $\frac{d^4y}{dx^4} = w_0$ (w_0 is constant; this relates to the displacements $y(x)$ of a bending beam).

Unit 2

1. Read the text and make a note of any useful word combination you find there

18.1 **Differential equations and their solutions**

Suppose that we have a problem in which a quantity x that we are studying depends on time t; that is to say, x is a function of t, which we write as $x(t)$. From the physics and geometry of the problem we can often obtain an indirect relation between x and y, called an equation for x. The equation might be an ordinary algebraic equation such as $x^2 + 2xt = 1$, but it might contain $\frac{dx}{dt}$ or $\frac{d^2x}{dt^2}$, as in the equation $\frac{d^2x}{dt^2} = g$ for a falling body, where g is the gravitational acceleration. This is simple example of a differential equation, and we can solve it by the methods of earlier chapters (compare problem 14.8f).

The equation

$$\frac{dx}{dt} = 3x$$

is also a differetial equation, but we do not yet know how to find an explicit solution for x in terms of t. Obviously not just anything will do; if for instance we try $x = t^2$ it does not work, because then $\frac{dx}{dt} = 2t$, but $3x = 3t^2$, and these are quite different.

A clue is given by interpreting the equation: it says that a quantity x always grows at a rate proportional to the amount of x already present. This is a property of the exponential function (see Section 1.10), so we might try exponential functions of t. In fact,

$$x = e^{3t}$$

solves the equation, because then $\frac{dx}{dt} = 3e^{3t}$, and this is equal to $3x$, as required. However, it is not the only solution, because

$$x = Ae^{3t},$$

where A is any constant, also solves the equation.

In general, a differential equation for x as a function of t is an equation involving at least the first derivative $\frac{dx}{dt}$ as well as, possibly, x and t separately. Some examples are

$$\frac{dx}{dt} + 2xt = 1, \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0, \frac{d^3x}{dt^3} = \frac{x^2}{t^2}.$$

In such equations, t is called the independ variable and x the depend variable. An equation is called first-order, second order and so on, according to the order of the highest derivative in it: $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, and so on.

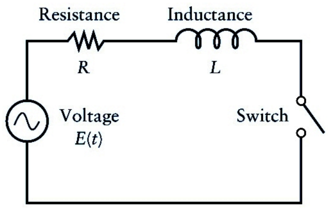

Fig. 18.1

Problems in science and engineering are often more easily formulated in terms of differential equations. Suppose for example that in the RL circuit of Fig. 18.1 the switch is closed at time $t = 0$, and that subsequently the voltage applied is $E(t)$. Then the current $x(t)$ is found by solving the differential equation

$$L\frac{dx}{dt} + Rx = E(t).$$

Here we have collected all the terms that involve x (including $\frac{dx}{dt}$) on the left side and have put the term that does not involve x, namely $E(t)$ on the right. This is the conventional arrangement. The term independent of x which comes on the right is then called forcing term, the reason being obvious in this case, since $E(t)$ drives the circuit.

The differential equation with the same left-hand side, but with a zero forcing term on the right, plays a key role in obtaining solutions of the original equation. Such equations are called unforced differential equations, or sometimes homogeneous equations, and are the subject of this chapter, Also, for the present, we shall further restict ourselves to linear equations with constant coefficients which have the form:

Linear differential equations with constant coefficients

- First-order:

$$\frac{dx}{dt} + cx = 0 \text{ } (c \text{ constant}).$$

- Second-order:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0 \text{ } (b,\, c \text{ constants}).$$

These are called linear because there are no squares, products, etc., involving x and its derivatives. Such equations have comparatively simple characteristics. The simplest instance of all is

$$\frac{dx}{dt} = 0.$$

It has solutions $x = A$, where A is any constant. There is therefore an infinity of solutions, and we must expect this to be true in more general cases too.

A solution of a differential equation is any function $x(t)$ which fits, or satisfies, the equation. This is illustrated in the next two examples.

2. Say it in English

1. величина, которую мы изучаем, зависит от
2. возможно, иначе говоря, в общем, однако, поскольку, тогда, по существу, например(2), также
3. обычное алгебраическое уравнение, такое, как
4. простой / самый простой пример дифференциального уравнения
5. расти со скоростью пропорциональной значению x
6. это не единственное решение
7. сформулировать в терминах дифференциальных уравнений
8. найти путем решения дифференциального уравнения
9. зависимая переменная, быть независимым от
10. это рассуждение очевидно, так как
11. далее мы ограничиваемся уравнениями вида

3. Find in the text you are going to read now a word which:

- means "*to check that sth is true or accurate by careful investigation*"
- means "*to need sth; to depend on sb / sth*"
- means "*to state or show that sth is definitely true or correct*"
- Example 18.1 *For a differential equation* $\frac{dx}{dt} + 2x = 0$, *verify that (a)* $x = e^{2t}$ *is not a solution, (b)* $x = 2e^{-2t}$ *is a solution.*

    1. Test $x = e^{2t}$. Then $\frac{dx}{dt} = 2e^{2t}$ and so
    $\frac{dx}{dt} + 2x = 2e^{2t} + 2e^{2t} = 4e^{2t}$.
    This is not zero, so $x = e^{2t}$ is not a solution.
    2. Test $x = 2e^{-2t}$. Then $\frac{dx}{dt} = -4e^{-2t}$ and so
    $\frac{dx}{dt} + 2x = -4e^{-2t} + 4e^{-2t} = 0$.
    The zero value is what the equation requires, so $2e^{-2t}$ is a solution.

Incidentally, we can confirm in the same way that $x = Ae^{-2t}$, where A is any constant, is always a solution. We have

$$\frac{dx}{dt} + 2x = -2Ae^{-2t} + 2Ae^{-2t} = 0,$$

as it should be. This is the infinity of solutions we were expecting.

4. Complete the text with the words and phrases in the box

therefore, verify, similarly, the two bracketed expressions, strightforward, not expected to show, as required, as follows, in this way

- Example 18.2 ... *that the following functions are solutions of the second-order equation* $\frac{d^2x}{dt^2} + 4x = 0$*: (a)* $x = \cos 2t$*, (b)* $x = \sin 2t$*, (c)* $x = A\cos 2t + B\sin 2t$*, where* A *and* B *are any constants.*

  Note that 'verify' means 'try out': you are ... how the solutions were obtained.

  1. If $x = \cos 2t$, then $\frac{dx}{dt} = -2\sin 2t$, and $\frac{d^2x}{dt^2} = -4\cos 2t$. ...
     $\frac{d^2x}{dt^2} + 4x = -4\cos 2t + 4\cos 2t = 0$
     as required.
  2. ... , if $x = \sin 2t$, then
     $\frac{d^2x}{dt^2} + 4x = -4\sin 2t + 4\sin 2t = 0$
  3. Confirmation is ..., but the underlying reason why the previous solutions can be combined into a new solution ... is made clearer by organizing the calculation ... .
     $\frac{d^2x}{dt^2} + 4x = \frac{d^2}{dt^2}(A\cos 2t + B\sin 2t) + 4(A\cos 2t + B\sin 2t)$
     $= A(\frac{d^2}{dt^2}\cos 2t + 4\cos 2t) + B(\frac{d^2}{dt^2}\sin 2t + 4\sin 2t) = 0$
     by rearranging the terms. We already know the ... are zero, so the whole expression is zero ... .

The separation of $\frac{d^2x}{dt^2} + 4x$ into an 'A' part and a 'B' part in this way is possible only because the equation is linear.

5. Read the text below. Explain the meaning of the word *constant* in which it occurs in the text and write down all the word combinations with this noun.

Solving first-order linear unforced equations

Consider the equation

$$\frac{dx}{dt} + cx = 0 \text{ (} c \text{ is a fixed constant).} \qquad (18.2)$$

If we write it in the form

$$\frac{dx}{dt} = (-c)x,$$

it can be seen to describe the variation of a quantity $x(t)$ which decays (if c is positive) or grows (if c is negative) at a rate proportional to the amount of x already present. From Section 1.10, we know that exponential functions have this property. We shall therefore test the solutions of the form

$$x(t) = Ae^{mt} \qquad (18.3)$$

where A and m are unknown constants which we shall try to adjust to fit the equation. From (18.3),

$$\frac{dx}{dt} + cx = Ame^{mt} + cAe^{mt} = A(m + c)e^{mt}.$$

This quantity must be zero for all values of t in order to fit the differential equation (18.2). Ignoring the possibility $A = 0$, which gives us the so-called trivial solution $x(t) = 0$, we must have

$$m = -c,$$

and in that case it does not matter what value is given to A. We have therefore found a collection of solutions $x(t) = Ae^{-ct}$, where A is an arbitrary constant. It can be proved that there are no other solutions, and so we call the solutions we have found the general solution of the equation.

The general solution of

$$\frac{dx}{dt} + cx = 0$$

where c is a given constant, is

$$x(t) = Ae^{-ct}, \tag{18.4}$$

where A is any constant.

- Example 18.3 *Find the general solution of* $\frac{dx}{dt} - 4x = 0$.

  We will rework the theory. Look for solutions of the form $x = Ae^{mt}$:

  $$\frac{dx}{dt} - 4x = Ame^{mt} - 4Ae^{mt} = Ae^{mt}(m-4).$$

  This is zero for all time if m = 4, whatever the value of A. Therefore the general solution (which includes the trivial one mentioned above) is $x = Ame^{4t}$, with A an arbitrary constant.

Figure 18.2 depicts several of these solutions, corresponding to various values of the arbitrary constant A.

Each value of A gives a different curve, and these solution curves fill the whole plane. Also the curves do not cross, so there is one and only one curve through every point. This corresponds to the fact that the slope $\frac{dx}{dt}$ has one and only one value at every point, namely the value prescribed by the differential equation $\frac{dx}{dt} = 4x$ taken at the point. This is all strong evidence that we have found all the solutions. More is said about the graphical way of understanding differential equations in Chapters 22 and 23.

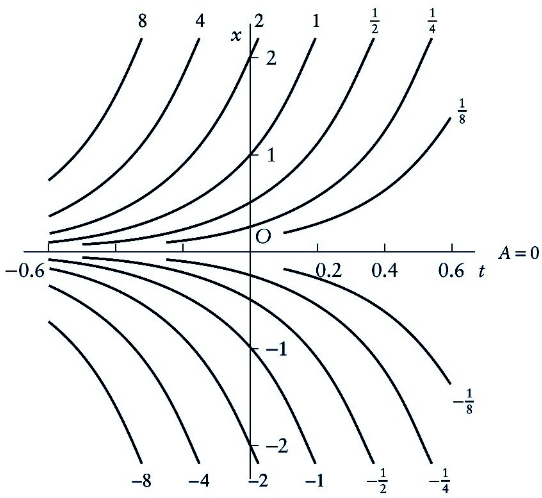

Fig. 18.2

- Example 18.4 *Find all the solutions of* $3\frac{dx}{dt} + 2x = 0$.

  We could carry out the full calculation as in the previous example. However, if instead we want to quote the formula, (18.4), we must first write the equation in the form

  $$\frac{dx}{dt} + \frac{2}{3}x = 0.$$

  Therefore $c = \frac{2}{3}$ (not 2), and the general solution is $x = Ae^{-\frac{2}{3}}$, with A any constant.

It is worth while to memorize the formula (18.4).

In practical cases we do not usually need all the solutions, but only the one which satisfies some further condition of the proflem. Frequently the condition supplied describes the condition prevailing at the start of the action, or at some other time, as in the following.

- Example 18.5 *Find the solution of* $\frac{dx}{dt} - 4x = 0$ *of which* $x = 2$ *when* $t = 1$.

  Other ways of saying this are 'find the solution curve which passes through the point $(1, 2)$', or 'find a solution $x(t)$ so that $x(1) = 2$'.

  From Example 18.3 all the possible solutions are given by

  $$x = Ae^{4t}.$$

  Since $x = 2$ when $t = 1$, we must have $2 = Ae^4$. Therefore

  $$A2e^{-4}$$

  and the single solution picked out is

  $$x = (2e^{-4})e^{4t} = 2e^{4(t-1)}.$$

An extra condition of this type is called an initial condition. It describes the state of the system at a given time. The differential equation together with its initial condition is called an initial-value problem.

Initial-value problem, first-order equation

(a) Differential equation:

$$\frac{dx}{dt} + cx = 0$$

(b) Initial equation:

$$x = x_0 \text{ at } t = t_0 \text{ or } (x(t_0) = x_0), \text{ with } x_0 \text{ and } t_0 \text{ specified.} \qquad (18.5)$$

6. Say it in English

1. Это соответствует тому факту, что ...
2. Решение, упомянутое выше
3. В этой связи стоит упомянуть тот факт, что ...
4. Чтобы найти решение, нужно выполнить вычисление, как в предыдущем примере.
5. На рисунке изображены некоторые из найденных решений.

7. In the text that follows, find word combinations with the noun *solution* and use some of them in the sentences of your own.

18.3 **Solving second-order linear unforced equations**

For second-order differential equations of the type (18.1b), we use a similar technique.

- Example 18.6 *Find some solutions of the equation*

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0.$$

We will look first for absolutely basic solutions. Test whether there are any solutions of the form $x(t) = e^{mt}$, where m is constant. Because $\frac{dx}{dt} = me^{mt}$ and $\frac{d^2x}{dt^2} = m^2e^{mt}$, we have

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = m^2e^{mt} + me^{mt} - 2e^{mt} = e^{mt}(m^2 + m - 2).$$

This is zero for all time if $m^2 + m - 2 = 0$, that is if

$$m = 1 \text{ or } -2.$$

This gives us two solutions, namely $x(t) = e^t$ and $x(t) = e^{-2t}$.

From this basis, we can obtain more solutions. Guided by Example 18.2c, we show that also

$$x(t) = Ae^t + Be^{-2t},$$

where A and B are arbitrary constants, is a solution. By substituting into the equation and sorting the terms into those with coefficient A and those with coefficient B, we obtain

$$A(\frac{d^2}{dt^2}e^t + \frac{d}{dt}e^t - 2e^t) + B(\frac{d^2}{dt^2}e^{-2t} + \frac{d}{dt}e^{-2t} - 2e^{-2t}) = 0,$$

because e^t and e^{-2t} are known already to be solutions; so both of the bracketed expressions are zero.

This is the principle, but consider now the general case

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0.$$

Look for solutions of the form $x = e^{mt}$. Then

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = e^{mt}(m^2 + bm + c).$$

This will be zero for all t, as required by the differential equation, if

$$m^2 + bm + c = 0, \tag{18.6}$$

which is called the characteristic equation. Being quadratic, it may have two real solutions, exactly one real solution, or two complex solutions depending on the coefficients. Consider the real cases first:

Roots m_1 and m_2 of the characteristic equation are real and different

In this case,

$$x(t) = e^{m_1 t} \text{ and } x(t) = e^{m_2 t}$$

are solutions of the differential equation, and from these we can construct a whole family of solutions

$$x(t) = Ae^{m_1 t} + Be^{m_2 t},$$

where A and B are arbitrary. It can be proved that there are no more solutions: this gives a basis for the general solution.

Characteristic equation: unequal real roots

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0; \text{ roots } m_1 \text{ and } m_2 \text{ of}$$

$$m^2 + bm + c = 0$$

real and different.

Basis of solutions:

$$e^{m_1 t}, e^{m_2 t}.$$

General solution:

$$Ae^{m_1 t} + Be^{m_2 t}, \ (A, \ B \text{ arbitrary}). \tag{18.7}$$

- Example 18.7 *Find the general solution of* $2\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = 0$.

  To correspond with the standard form, (18.7), we should have to write the equation in the form $\frac{d^2x}{dt^2} - \frac{1}{2}\frac{dx}{dt} - \frac{1}{2}x = 0$, but there is no need to do this if we directly test for solutions of the form $x = e^{mt}$. The characteristic equation then takes the form $2m^2 - m - 1 = 0$, or $(2m+1)(m-1) = 0$, so that $m_1 = -\frac{1}{2}$, $m_2 = 1$. Therefore the basis for the general solution in the solution pair $(e^{-\frac{1}{2}t}, e^t)$, and the general solution is $x(t) = Ae^{-\frac{1}{2}t} + Be^t$, A and B arbitrary.

Roots m_1 and m_2 of the characteristic equation are equal

Suppose that $m_1 = m_2 = m_0$, say. We have then only one function for our basis instead of two, and we might expect the general solution to be $Ae^{m_0 t}$. However, all we know is that there is essentially only one solution of the form e^{mt} (ignoring simple multiples of e^{mt}), but we shall see in the next example that there is also a solution which is not of this form, namely

$$x(t) = te^{m_0 t}.$$

We might therefore think there will be no end of it: if $te^{m_0 t}$ is a solution, then why not $t^2 e^{m_0 t}$, or some function of great complication? However, it can be proved that every second-order linear equation has exactly the linearly independent solutions (i.e. they are not just multiples of each other); also that these form a basis of solutions: we do not need any others to construct the most general solution. Formally:

Basis and general solution of

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

(a) There exist two linearly independent solutions.

(b) If $u(t)$ and $v(t)$ are any two linearly independent solutions, these form basis for general solution; that is to say, the general solution is given by

$$x(t) = Au(t) + Bv(t),$$

where A and B are arbitrary constants.

- Example 18.8 *Find the general solution of* $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0.$

  The charactiristic equation, formed by substituting $x(t) = e^{mt}$ is

$$m^2 + 4m + 4 = (m + 2)^2 = 0,$$

  and the only solution of m that we find is $m = -2$. It corresponds to the basic solution e^{-2t}.

  The theorem (18.9) guarantees there is another independent solution and it does not matter how we find it. Test the truth of (18.8), which proposes an independent solution having the form

$$x(t) = te^{-2t}.$$

  Then

$$\frac{dx}{dt} = (1 - 2t)e^{-2t},$$

  and

$$\frac{d^2x}{dt^2} = (-4 + 4t)e^{-2t}.$$

  Therefore

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = [(-4 + 4t) + 4(1 - 2t) + 4t]e^{-2t},$$

  which is zero, so $x(t) = te^{-2t}$ is a second solution , and it is independent of the first. By (18.9), the solution basis is therefore

$$(e^{-2t}, te^{-2t}),$$

  and the general solution is $x(t) = Ae^{-2t} + Bte^{-2t}$, A and B arbitrary.

The second solution always takes the same form (see Problem 18.8):

Characteristic equation: coincedent roots

If $\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, in which $b^2 - 4c = 4$ (for coincedent roots), and m_0 is the single solution of the characteristic equation $m^2 + bm + c = 0$, then the solution basis is (e^{m_0t}, te^{m_0t}) and the general solution is $x(t) = Ae^{m_0t} + Be^{m_0t}$ (A and B are arbitrary constants).

8. The verb *substitute* means "to take place of sb/sth else; to use sb/sth instead of sb/sth else." We can use it and its derivatives as in the examples below. Study them.

1. This axiom claims that equivalent descriptions (e.g., "a" and "a'") can be substituted for one another.
2. We substitute knowledge for reasoning.
3. In particular, a difference scheme can be examined for stability by substituting into it perturbed values of the solution.
4. This alloy is used as a substitute for silver.

Translate into English

1. Подставляя (1) в уравнение (2), мы получаем ...
2. Скорость может быть найдена подстановкой (2.3) в (2.4).
3. В соответствии с условием задачи эти величины взаимозаменяемы.

9. Try to recollect how these phrases are used in the text you have just read.

There is essentially only one solution of ...; we can construct a whole family of solutions; but there is no need to do this; being quadratic; guided by Example 18.2c; it can be proved that ...; by substituting into ...; to correspond with the standard form; as required by the differential equation; it corresponds to the basic solution; by substituting $x(t) = e^{mt}$; it is independent of ... .

10. Fill in the missing prepositions.

18.4 **Complex roots of the characteristic equation**
If $b^2 < 4c$, the roots m_1 and m_2 of the characteristic equation m^2+bm+c for the differential equation $\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ are complex. Since they are roots ... a quadratic equation, they must be complex conjugate, so put

$$m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta,$$

where α and β are real numbers. The corresponding functions

$$e^{(\alpha+i\beta)t} \text{ and } e^{(\alpha-i\beta)t}$$

are genuine complex solutions, so we call (18.11) a complex basis ... solutions of the differential equation. If we are interested ... complex as well as real solutions, then we can allow the arbitrary constants A and B to be complex as well, in an all-inclusive general complex solution

$$x(t) = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}.$$

Suppose, however, that we want the general solution to consist only ... real functions, then a basis for a real solutions can be got from (18.11) ... the following way. By (6.8)

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t.$$

This function solves the differential equation, so its real and imaginary parts separately must also solve it, Therefore

$$(e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t)$$

is a real basis ... the general (real) solution

$$x(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t, \tag{18.12}$$

where A and B are arbitrary (but real, of course). The second complex solution, $e^{(\alpha-i\beta)t}$, has the basis, $(e^{\alpha t}\cos\beta t, -e^{\alpha t}\sin\beta t)$ which leads ... the same family of solutions, so we get nothing new ... considering it.

Equation (18.12) can be written ... a different form. Using the identity (1.19), we have

$$A\cos\beta t + B\sin\beta t = A\cos\beta t + \phi,$$

where C and ϕ are constants related ... A and B. Therefore (18.12) can be written

$$x(t) = Ce^{\alpha t}\cos\beta t + \phi.$$

Since A and B are arbitrary, so are C and ϕ.

11. Complete the text with the word combinations in the box.

the real solutions, the general solution, the complex solution basis

- Example 18.9 *Find the general solution of*

$$\frac{d^2x}{dt^2} + 4x = 0.$$

The characteristic equation is $m^2 + 4 = 0$. Its solutions are $m = \pm 2i$. Therefore ... is (e^{2it}, e^{-2it}). But

$$e^{2it} = \cos 2t + i\sin 2t,$$

and the real and imaginary parts give a basis for ...:

$$(\cos 2t, \sin 2t).$$

Therefore ... is

$$x(t) = A\cos 2t + B\sin 2t \ (A,\ B \text{ arbitrary}).$$

12. Give the English equivalents of the Russian words and phrases in brackets:

- Example 18.10 *(Найдите общее решение следующего уравнения)*

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0.$$

Setting $x = e^{mt}$ (дает характеристическое уравнение) $m^2 + 2m + 2 = 0$, so that $m = -1 \pm i$. Therefore

$$(e^{(-1+i)t}, e^{(-1-i)t})$$

is a basis for complex solutions. But

$$e^{(-1+i)t} = e^{-t}(\cos t + i \sin t),$$

(где действительной и мнимой частями являются)

$$e^{-t} \cos t, e^{-t} \sin t.$$

(Они формируют основу) for the real solutions. The general solution is

$$x(t) = Ae^{-t} \cos t + Be^{-t} \sin t.$$

If we chose instead to take the real and imaginary parts of $e^{(-1+i)t}$, (мы бы получили) $(e^{-t} \cos t, -e^{-t} \sin t)$ (в качестве базиса). The minus sign will be absorbed into the arbitrary constant B: (никакое новое решение не возникает).

13. Go on reading the text and make notes of any useful word combination you find there.

The general solution method can be summed up as follows:
Characteristic equation: complex roots
$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, when $m^2 + bm + c$ has complex roots $m_1, m_2 = \alpha \pm i\beta$ (i.e. $b^2 < 4c$).
Complex basis: $e^{(\alpha+i\beta)t}$, $e^{(\alpha-i\beta)t}$.
Real basis: $e^{\alpha t} \cos \beta t$, $e^{\alpha t} \sin \beta t$.
General solution:
(a)

$$x(t) = Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t$$

(A and B arbitrary);
or
(b)

$$x(t) = Ae^{\alpha t} \cos \beta t + \phi \tag{18.13}$$

(C and ϕ arbitrary).
A very important case is when $b = 0$ and $c > 0$, illustrated by Example 18.9. In that case, $\alpha = 0$. In conventional notation, putting $c = \omega^2$, we obtain the following result:
Characteristic equation: special case

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Characteristic equation: $m^2 + \omega^2 = 0$; $m1, m2 = \pm i\omega$.
Complex basis: $e^{i\omega t}$, $e^{-i\omega t}$.
Real basis: $\cos \omega t$, $\sin \omega t$.
General solution":
(a)

$$x(t) = A \cos \omega t + B \sin \omega t,$$

or
(b)

$$x(t) = C \cos (\omega t + \phi) \tag{18.14}$$

In the special case (18.14), the alternative solution form

$$x(t) = C\cos(\omega t + \phi)$$

shows that the solutions oscillate regularly, swinging above and below the t axis to an extent governed by the amplitude C. In the general case (18.13),

$$x(t) = Ce^{\alpha t}\cos(\beta t + \phi),$$

the solutions oscillate, but the amplitude is governed by the factor $Ce^{\alpha t}$. If α is positive, the oscillation constantly grows; if α is negative, it dies away to zero. This is fully discussed in Chapter 20, but Fig. 18.3 shows a particular case where α is negative.

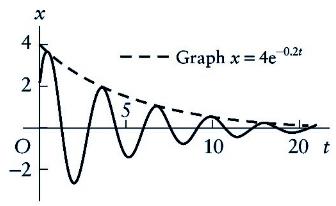

Fig. 18.3

The damped unforced linear oscillator is the simplest linear model of an oscillating mechanical or electrical system which has a small amount of friction or some other form of energy-loss mechanism (see Chapter 20 for a full discussion). In a customary notation the equation is

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0.$$

The term $2k\frac{dx}{dt}$ expresses the energy-absorbing property. Assume

$$k^2 < \omega^2.$$

The characteristic equation is $m^2 + 2km + \omega^2 = 0$, so that

$$m = -k \pm (k^2 - \omega^2)^{\frac{1}{2}} = -k \pm i(\omega^2 - k^2)^{\frac{1}{2}},$$

since $k^2 < \omega^2$. From (18.13), $\alpha = -k$ and $\beta = (\omega^2 - k^2)^{\frac{1}{2}}$, so finally:

Damped Linear Oscillator

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0 \text{ where } k^2 < \omega^2$$

General solutions:

(a)

$$x(t) = Ae^{-kt}\cos[(\omega^2 - k^2)^{\frac{1}{2}}t + \phi]$$

(A and B arbitrary constants); or

(b)

$$x(t) = Ce^{-kt}\cos(\omega^2 - k^2)^{\frac{1}{2}}t + \phi \tag{18.15}$$

(C and ϕ arbitrary).

18.5 **Initial conditions for second-order equations**

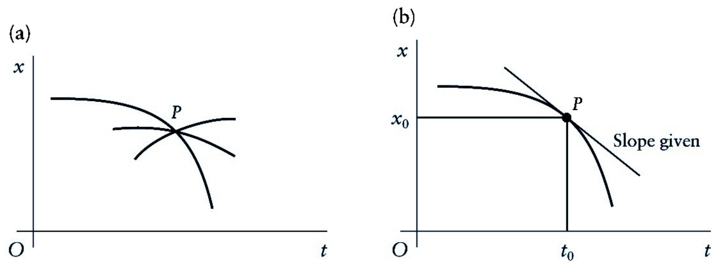

Fig. 18.4

The general solution of a second-order differential equation involves two arbitrary constants, and the solutions are therefore an order of magnitude more numerous than in the first-order case. Unlike the first-order case, the solution curves may cross – in fact, there is an infinite number of solution curves through any point on the (x, t) plane, as indicated in Fug. 18.4a.

To pick out a particular solution, we need to determine the two arbitrary constants. Two pieces of information are necessary. These may consist of two initial conditions, conditions which define the state of the system at some starting time t_0: the value of $x(t)$ and the slope $\frac{dx}{dt}$ at $t = t_0$ are given (see Fig. 18.4b). For example, the equation $\frac{d^2x}{dt^2} + \omega_0^2 x = 0$ describes the oscillations of a particle on a spring; the initial conditions tell us its position and velocity (i.e. its state) when it starts off. We then have an initial-value problem:

Initial-value problem

(a) Equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0,$$

(b) Initial conditions:

$$x = x_0 \text{ and } \frac{dx}{dt} = x_1 \text{ at } t = t_0,$$

which may be expressed alternatively as

$$x(t_0) = x_0,\ x'(t_0) = x_1,$$

where x_0 and x_1 are given.

- Example 18.11 *Find the solution of* $\frac{d^2x}{dt^2} + 4x = 0$ *for which* $x = 1$ *and* $\frac{dx}{dt} = 2$ *at* $t = 0$ *(i.e.* $x(0) = 1$, $x'(0) = 2$*).*

  First we need all the solutions. From Example 18.9, these are $x(t) = A\cos xt + B\sin 2t$, where A and B may take any values. Since $x = 1$ at $t = 0$,

  $$1 = A + 0, \text{ so } A = 1.$$

  For the other condition, we first need $x'(t)$ in general:

  $$x'(t) = -2A\sin 2t + 2B\cos 2t.$$

  At $t = 0$, we are given that $x'(0) = 2$, so the last equation becomes

  $$2 = 0 + 2b, \text{ or } B = 1.$$

  The required solution is therefore $x(t) = \cos 2t + \sin 2t$.

14. Use the words and phrases from what you have read to solve the following problems.

Problems

18.1 Say which of the following equations are linear, unforced, with constant coefficients (i.e. can be rearranged to conform with (18.1a)),

(a) $x' = 3t$;

(b) $x' = \frac{1}{2}t$;

(c) $x' + tx = 0$;

(d) $3x' - 2x^2 = 0$;

(e) $x' - x = 0$;

(f) $x' = 0$;

(g) $\frac{x'}{x^2} = 3$;

(h) $\frac{dy}{dx} + \frac{1}{2}y = 1$;

(i) $\frac{1}{y}\frac{dy}{dx} = 2$;

(j) $L\frac{dI}{dt} + RI = 0$;

(k) $\frac{v'+v+v^2}{v'-v+v^2} = 1$.

18.2 Write down all the solutions of the following equatons. Check one or two of them by substitution into the differential equation.

(a) $x' + 5x = 0$;

(b) $x' - \frac{1}{2}x = 0$;

(c) $x' - x = 0$;

(d) $x' + 3x = 0$;

(e) $3x' + 4x = 0$;

(f) $x' = 2x$;

(g) $x' = 3x$;

(h) $\frac{x'}{x} = -3$;

(i) $\frac{x'+1}{x+1} = 1$.

18.3 Solve the following initial-value problems.

(a) $x' + 2x = 0$, $x = 3$ when $t = 0$;

(b) $3x' - x = 0$, $x = 1$ when $t = 1$;

(c) $y' - 2y = 0$, $y = 2$ when $x = -3$;

(d) $x' + x = 0$, $x(-1) = 10$;

(e) $2y' - 3y = 0$, $y(0) = 1$;

(f) Find the curve whose slope at any point (x, y) is equal to $5x$, and which passes through the point $(1, -2)$.

18.4 Suppose that the generator in Fig 18.1 is short-circuited and cut out at the moment when the current in the circuit is I_0. Find an expression for the current subsequently. Show that the ratio $\frac{L}{r}$ provides a measure of the time it takes for the current to die away.

18.5 A radioactive element disintegrates at a rate proportional to the amount of the original element still remaining. Show that if $A(t)$ represents the activity of the element at time t, then

$\frac{dA}{dt} + kA = 0,$

where k is a positive constant.

(a) Solve the initial-value problem for A if $A = A_0$ (given) at time $t = 0$.

(b) The time taken for activity to drop to half of if the starting value is called the half-life period. For uranium-232, it is found that 17.5% has decayed after 20 years. Show that its half-life period is about 72 years.

18.6 Once upon a time, rabbits in Elysium reached maturity instantly and bred with a birthrate of 20 rabbits per year per couple. No rabbit ever died. At the start of the experiment Zeus released 50 male and 50 female rabbits.

By treating the number of rabbits as a continuously varying quantity and considering the number born in a short time δt, construct a differential equation and then an initial-value problem for $R(t)$, the rabbit population. Find how many rabbits there were at the end of Year 4.

Appaled by this result and assisted by Pluto, Zeus launched another similar experiment, in which any rabbit was allowed to live one year only. Construct the differential equation for the population. Did this alleviate the situation appreciably?

18.7 Obtain all solutions of the following equations. (The characteristic equations all have real roots, not nesessarily distinct.)

(a) $x'' - 3x' + 2x = 0;$

(b) $x'' + x' - 2x = 0;$

(c) $x'' - x = 0;$

(d) $x'' - 4x = 0;$

(e) $3x'' - \frac{1}{4}x = 0;$

(f) $x'' - 9x = 0;$

(g) $x'' + 2x' - x = 0;$

(h) $x'' - 2x' - 2x = 0;$

(i) $2x'' + 2x' - x = 0;$

(j) $3x'' - x' - 2x = 0;$

(k) $x'' + 4x' + 4x = 0;$

(l) $x'' + 6x' + 9x = 0$;

(m) $4x'' + 4x' + x = 0$;

(n) $x'' = 0$.

18.8 Verify that, when the characteristic equation corresponding to $x'' + bx' + cx = 0$ has coincident roots $m_1 = m_2 = m_0$, say, then the function $x(t) = t^{m_0 t}$ provides a second solution for the basis of the general solution. (For coincident roots, $b^2 = 4c$.)

18.9 Solve the following initial-value problems.

(a) $x'' - 4x = 0$, $x(0) = 1$, $x'(0) = 0$;

(b) $x'' + x' - 2x = 0$, $x(0) = 0$, $x'(0) = 2$;

(c) $y'' - 4y' + 4Y = 0$, $y(0) = 0$, $y'(0) = -1$;

(d) $y'' + 2y' + y = 0$, $y(1) = 0$, $y'(1) = 1$;

(e) $x'' - 9x = 0$, $x(1) = 1$, $x'(1) = 1$;

(f) $x'' - 4x = 0$, $x(1) = 1$, $x'(1) = 0$.

18.10 Obtain all the solutions of the following equations. (The roots of the characteristic equations are complex.)

(a) $x'' + x = 0$;

(b) $x'' + 9x = 0$;

(c) $x'' + \frac{1}{4}x = 0$;

(d) $x'' + \omega_0^2 x = 0$;

(e) $x'' + 2x' + 2x = 0$;

(f) $y'' - 2y' + 2y = 0$;

(g) $y'' + y' + y = 0$;

(h) $2x'' + 2x' + x = 0$;

(i) $3x'' + 4x' + 2x = 0$;

(j) $3x'' - 4x' + 2x = 0$.

18.11 Solve the following initial-value problems.

(a) $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$;

(b) $x'' + 4x = 0$, $x(0) = 1$, $x'(0) = 0$;

(c) $x'' + \omega_0^2 x = 0$, $x(0) = a$, $x'(0) = b$;

(d) $x'' + 2kx' + x = 0$, $x(0) = 0$, $x'(0) = b$ for the cases $k^2 > 1$, $k^2 < 1$, and $k^2 = 1$.

(Use the A, B form: finding the constant C and ϕ in (18.14b) for an initial-value problem can be comparatively difficult.)

18.12 The approximate equation for small swings of a pendulum is

$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$

where θ is the inclination from the vertical (in radians), l is the length, and g is the gravitational acceleration. The pendulum is held still at an angle α, and is then passively released. Find the subsequent motion.

18.13 The pendulum in Problem 18.12 is hanging at rest; then the bob is given a small velocity v in the direction of θ increasing. Find the subsequent motion.

18.14 If there is a little friction in the pendulum of Problem 18.12, the equation of motion takes the form

$\frac{d^2\theta}{dt^2} + K\frac{d\theta}{dt} + \frac{g}{l}\theta = 0,$

where K is an additional positive constant which takes account of the friction (assumed to be proportional to the angular velocity). In a particular case (SI unit), $g = 9.7$, $l = 20$, $K = 0.066$. The pendulum is at rest at first, hanging freely. It is then pushed so as to give the bob a velocity of 1 metre per second. Find the subsequent motion.

18.15 Consider the third-order differential equation

$\frac{d^3y}{dx^3} - y = 0.$

Proceed by analogy with the method of Section 18.3 by substituting $y = e^{mx}$, and obtaining a characteristic equation for m (a cubic equation), find three distinct basic solutions of this type. By intoducing arbitrary constants A, B, C, find as wide a variety of solutions as you can (in fact, this is the general solution).

18.16 By proceeding as in Problem 18.15, find a wide variety of solutions of the equation

$\frac{d^3y}{dx^3} + y = 0.$

18.17 By proceeding with the equation

$\frac{d^4y}{dx^4} - y = 0$

as in Problem 18.15, obtain the collection of solutions

$y(x) = Ae^x + Be^{-x} + C\cos x + D\sin x,$

where A, B, C, D are arbitrary constants.

18.18 A tapered concrete column of height H metres is to support a statue of mass M (i.e. weight Mg force units, where g is the gravitational acceleration) at the top. Pressure (force per unit area) may not exceed P. Show that the most economical construction for the column is for its cross-sectional area $A(y)$, where y is distance above the ground, to satisfy the equation

$A(y) = \frac{Mg}{P} + \frac{\rho g}{P}\int_y^H A(u)du,$

where ρ is the density of concrete. By differentiating this expression, obtain a differential equation for $A(y)$, and an initial condition for the equation, and solve it.