

Equations and Inequalities: Elementary Problems and Theorems in Algebra and Number Theory

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abridged for academic purposes*

The present book is technique-oriented. It explains how large classes of problems may be attacked, and the problems themselves are much more like the exercises found in texts.

The book is divided into four parts: Algebraic Identities and Equations; Algebraic Inequalities; Number Theory; and Hints and Answers. Each chapter presents an assortment of techniques for dealing with the problems in that area.

Chapter 1 discusses finite sums and combinatorial identities. As might be suspected, for many such sums the imaginative use of the binomial theorem is sufficient to prove the formula in question. One unusual aspect of this chapter is the attention devoted to sums of the form

$$S(q, n) = P(1)q + P(2)q^2 + P(3)q^3 + \dots + P(n)q^n,$$

where $P(x)$ is a polynomial of degree m . The authors state (without proof) a valuable transformation theorem: if $q \neq 1$ is real, then there exists a polynomial $Q(x)$ of degree m and a real number d such that

$$S(q, n) = d + Q(n)q^n.$$

The authors show how the polynomial Q may be found by the method of undetermined coefficients.

There is a valuable discussion of polynomials and their properties. One common practice in proving an identity is to show the quantity in question satisfies a polynomial equation with one or more known roots, and to rule out the presence of the other roots. For instance, to prove

$$\sqrt[3]{3\sqrt{21} + 8} - \sqrt[3]{3\sqrt{21} - 8} = 1,$$

denote the left hand side by A . Elementary algebra gives

$$A^3 + 15A - 16 = 0.$$

Thus A is a zero of the polynomial $F(x) = x^3 + 15x - 16$. $x = 1$ is also a zero of this polynomial. We find $F(x)/(x - 1) = x^2 + x + 16$, a polynomial which has only complex roots. Thus $A = 1$, as required.

The discussion of symmetric polynomials is unusual. The following result, which has generalizations to more than two variables, is invaluable: *for an arbitrary polynomial $F(x_1, x_2)$ symmetric in the variables x_1 and x_2 there exists a unique polynomial $H(y_1, y_2)$ such that*

$$F(x_1, x_2) = H(\sigma_1, \sigma_2)$$

$$\sigma_1 = x_1 + x_2, \quad \sigma_2 = x_1 x_2$$

(The proof of this and its extensions is in van der Waerden's algebra book.) The authors give a table expressing powers $x_1^n + x_2^n$ in terms of σ_1 and σ_2 for $n = 1, 2, \dots, 5$, and a similar table for the case of three variables. Symmetric functions are very useful in solving simultaneous nonlinear algebraic equations when the equations are symmetric in the variables. As an example, consider the simultaneous equations:

$$x_1^5 + x_2^5 = 464$$

$$x_1 + x_2 = 4.$$

One can eliminate one of the variables, but then one is faced with the problem of solving a quartic equation. Rewrite the equations using symmetric functions:

$$\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 = 464$$

$$\sigma_1 = 4,$$

This leads rapidly to the solutions $x_1 = 2 \pm \sqrt{2}$, $x_2 = 2 \mp \sqrt{2}$. (It turns out that MAPLE can solve the original equations, but it is not difficult to devise similar equations solvable this way which will cripple computer algebra systems.)

Section 6 of this Chapter deals with the solution of irrational algebraic equations, and introduces a variety of techniques. A typical example of the arcane equations considered here is

$$\sqrt[3]{x+1} + \sqrt[3]{x+2} + \sqrt[3]{x+3} = 1.$$

As might be expected, computer algebra systems fail on these sorts of equations.

Chapter 2, Algebraic Inequalities, is one of the most illuminating studies of the subject. Again, symmetry becomes a powerful tool. If one is attempting to show that

$$f(x_1, x_2, \dots, x_n) > 0$$

and f is symmetric in all its variables, then it is no loss of generality to assume $x_1 \geq x_2 \geq \dots \geq x_n$. This assumption can have a great efficacy. For example, suppose we wish to show that

$$(a+b+c)^{100} < 3^{100}(a^{100} + b^{100} + c^{100})$$

for a, b, c positive. Assume $a \geq b \geq c$. We get $(a+b+c)^{100} \leq (3a)^{100} < 3^{100}(a^{100} + b^{100} + c^{100})$.

The same technique will work to prove that

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0$$

for positive a, b, c , since the left-hand side above may be written

$$a(a-b)(a-c) + (c-b)[c(c-a) - b(b-a)],$$

and the assumption that $0 < a \leq b \leq c$ guarantees that both the term in brackets and the first term are positive.

As an exercise, the authors ask the reader to demonstrate an intriguing homogeneous nonnegative lower bound for the difference between the arithmetic and the geometric mean:

$$\frac{a+b}{2} - \sqrt{ab} \geq \frac{(a-b)^2(a+3b)(b+3a)}{8(a+b)(a^2+6ab+b^2)}.$$

Inequalities between means and generalizations of Cauchy's inequality receive a sound treatment also.

The third chapter, Number Theory, is self-contained, with a treatment of basic concepts such as divisibility, the Euclidean algorithm, least common multiple and greatest common divisor, prime numbers, primality, Euler's ϕ -function. The number of results that may be proved using very elementary techniques is apparently boundless. Here are just a few of them:

i) Any prime number of the form $2^{2^n} + 1$ cannot be expressed as a difference of fifth powers of two natural numbers.

ii) Let a, b , and $\sqrt{a} + \sqrt{b}$ be positive rational. Then \sqrt{a} and \sqrt{b} are rational.

iii) The number $2^{60} + 7^{30}$ is divisible by 13.

Congruences receive a thorough treatment, including systems of congruences and nonlinear congruences. The discussion of diophantine equations is extensive and systematic, and the equations treated include many strange nonlinear diophantine equations, for instance

$$1 + x + x^2 + x^3 = 2^y.$$

(SOLUTIONS: $x = 1, y = 2$, or $x = 0, y = 0$). The chapter closes with a discussion of the factorizability of polynomials and a statement of the Eisenstein irreducibility criterion.

The fourth chapter has the solutions for the exercises. Sometimes these are only sketched, which is appropriate.