Московский государственный университет
имени М. В. Ломоносова
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Практикум по чтению литературы по специальности для студентов-механиков и математиков

Учебное пособие

Москва
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Данное пособие направлено на развитие навыков чтения у студентов механико-математического факультета. Материалом для пособия послужили аутентичные статьи из журналов *The Mathematical Intelligencer*, *Scientific American* и др., дополненные лексико-грамматическими упражнениями. Помимо этого в пособии содержатся задания, цель которых — совершенствование навыков реферирования и перевода.

Пособие предназначено для самостоятельной работы студентов.

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I. The Greek View of Motion

Isaak Asimov, Understanding Physics, First Mentor Printing, April, 1969.

Among the first phenomena considered by the curious Greeks was motion. *One might initially suspect that motion is an attribute of life; after all, men and cats move freely but corpses and stones do not.* A stone can be made to move, to be sure, but usually through the impulse given it by a living thing.

However, this initial notion does not stand up, for there are many examples of motion that do not involve life. Thus, the heavenly objects move across the sky and the wind blows as it wills. Of course, it might be suggested that heavenly bodies are pushed by angels and that wind is the breath of a storm-god, and indeed such explanations were common among most societies and through most centuries. The Greek philosophers, however, were committed to explanations that involved only that portion of the universe that could be deduced by human reason from phenomena apparent to human senses. That excluded angels and storm-gods.

Furthermore, there were pettier examples of motion. The smoke of a fire drifted irregularly upward. A stone released in midair promptly moved downward, although no impulse in that direction was given it. Surely not even the most mystically minded individual was ready to suppose that every wisp of smoke, every falling scrap of material, contained a little god or demon pushing it here and there.

The Greek notions on the matter were put into sophisticated form by the philosopher Aristotle (384-322 B.C.). He maintained that each of the various fundamental kinds of matter ("elements") had its own natural place in the universe. The element "earth", in which was included all the common solid materials about us, had as its natural place the center of the universe. All the earthy matter of the universe collected then and formed the world upon which we live. *If every portion of the earthy material got as close to the center as it possibly could, the earth would have to take on the shape of a sphere (and this, indeed, was one of several lines of reasoning used by Aristotle to demonstrate that the earth was spherical and not flat).*

The element "water" had its natural plan about the rim of the sphere of "earth." The element "air" had its natural plan about the rim of the sphere of "water" and the element "fire" had its natural place outside the sphere of "air."

While one can deduce almost any sort of scheme of the universe by reason alone, it is usually felt that such a scheme is not worth spending time on unless it corresponds to "reality" – to what our senses tell us about the universe. In this case, observations seem to back up the Aristotelian view. As far as the senses can tell, the earth is indeed at the center of the universe; oceans of water cover large portions of the earth; the air extends about land and sea; and in the airy heights there are even occasional evidence of a sphere of fire that makes itself visible during storms in the form of lightning.

The notion that every form of substance has its natural plan in the universe is an example of an assumption. *It is something accepted without proof, and it is incorrect to speak of an assumption as either true or false, since there is no way of proving it to be either. (If there were, it would no longer be an assumption.) It is better to consider assumptions as either useful or useless, depending on whether or not deductions made from them corresponded to reality.*

If two different assumptions, or sets of assumptions, both lead to deductions that correspond to reality, then the one that explains more is the more useful.

On the other hand, it seems obvious that assumptions are the weak points in any argument, as they have to be accepted on faith in a philosophy of science that prides itself on its rationalism. Since we must start somewhere, we must have assumptions, but at least let us have as few assumptions as possible. Therefore, of two theories that explain equal areas of the universe, the one that begins with fewer assumptions is the more useful. Because William of Ockham (1300? -1349?), a medieval English philosopher, emphasized this point of view, the effort made to whittle away at unnecessary assumptions is referred to as making use of "Ockham's razor".

The assumption of "natural plan" certainly seemed a useful one to the Greeks. *Granted that such a natural place existed, it seemed only reasonable to suppose that whenever an object found itself out of its natural place, it would return to that natural place as soon as given the chance.* A stone, held in the hand in midair, for instance, gives evidence of its "eagerness" to return to its natural place by the manner in which it presses downwards. This, one might deduce, is why it has weight. If the supporting hand is removed, the arm promptly moves toward its natural place and falls downward. By the same reasoning, we can explain why tongues of fire shoot upward, why pebbles fall down through water, and why bubbles of air rise up through water.

One might even use the same line of argument to explain rainfall. When the heat of the sun vaporizes water ("turns it into air" a Greek might suppose), the vapors promptly rise in search of their natural place. *Once those vapors are converted into liquid water again, the latter falls in droplets in search of their natural place.*

From the assumption of "natural place," further deductions can be made. One object is known to be heavier than another. *The heavier object pushes downward against the hand with a greater "eagerness" than the lighter object does.* Surely, if each is released the heavier object will express its greater eagerness to return to its place by falling more rapidly than the lighter object. *So Aristotle* *maintained, and indeed this too seemed to match observation, for light objects such as feathers, leaves, and snowflakes drifted down slowly, while rocks and bricks fell rapidly.*

But can the theory withstand the test of difficulties deliberately raised? For instance, an object can be forced to move away from its natural place, as when a stone is thrown into the air. This is initially brought about by muscular impulse, but once the stone leaves the hand, the hand is no longer exerting an impulse upon it. Why then doesn't the stone at once resume its natural motion and fall to earth? Why does it continue to rise in the air?

Aristotle's explanation was that the impulse given the stone was transmitted to the air and that the air carried the stone along. As the impulse was transmitted from point to point in the air, however, it weakened and the natural motion of the stone asserted itself more and more strongly. Upward movement slowed and eventually turned into a downward movement until finally the stone rested on the ground once more. Not all the force of the arm or a catapult could, in the long run, overcome the stone's natural motion. ("Whatever goes up must come down" we still say.)

It therefore follows that forced motion (away from the natural place) must inevitably give way to natural motion (toward the natural place) and that natural motion will eventually bring the object to its natural place. Once there, since it has no place else to go, it will stop moving. The state of rest, or lack of motion is therefore the natural state.

This, too, seems to square with observation, for thrown objects come to the ground eventually and stop; rolling or sliding objects eventually come to a halt; and even living objects cannot move forever. If we climb a mountain, we do so with an effort, and as the impulse within our muscles fades, we are forced to rest at intervals. Even the quietest motions are at some cost, and the impulse within every living thing eventually spends itself. The living organism dies and returns to the natural state of rest. ("All men are mortal.")

But what about the heavenly bodies? The situation with respect to them seems quite different from that with respect to objects on earth. For one thing, whereas the natural motion of objects here below is either upwards or downward the heavenly bodies neither approach nor recede but seem to move in circles about the earth.

Aristotle could only conclude that the heavens and the heavenly bodies were made of a substance that was neither earth, water, air, nor fire. *It was, a fifth "element," which he named "ether" (a Greek word meaning "blazing", the heavenly bodies being notable for the light they emitted).*

The natural place of the fifth element was outside the sphere of fire. *Why then, since they were in their natural place, did the heavenly bodies not remain at rest?* Some scholars eventually answered that question by supposing the various heavenly bodies to be in the charge of angels who perpetually rolled them around the heavens, but Aristotle could not indulge in such easy explanations. *Instead, he was forced into a new assumption to the effect that the laws governing the motion of heavenly bodies were different from those governing the motion of earthly bodies.* Here the natural suite was rest, but in the heavens the natural state was perpetual circular motion.

Exercises

1. *In the text, translate the sentences italicized.*

2. *In 4-6 sentences express Aristotle's view on motion.*

3. *Insert prepositions:*

to move ... the sky, to be committed ... smth, apparent ... smb, to put ... a form, to depend ... smth, to correspond ... smth, ... the other hand, to lead ... deductions, to refer ... smth, ... the same reasoning, to convert ... smth, ... instance, to exert an impulse ... smth, to transmit smth ... smth, ... the long run, to give way ... smth, to remain ... rest.

4. *Give the English equivalents of:*

1) явление, рассмотренное кем-либо
2) небесные тела
3) предположить, что
4) понятие о материи

5) твердые тела
6) заслуживающий рассмотрения
7) в этом случае
8) делать умозаключения из
9) кажется разумным предположить, что
10) рассуждать подобным образом
11) соответствовать наблюдениям
12) ни приближаться, ни удаляться относительно чего-либо

5. *Look up the following words in your English-English dictionary and check their pronunciation and stress:*

furthermore, sphere, scheme, assume, assumption, muscle, muscular, climb, scholar, suit, suite.

6. *Use a monolingual dictionary to check the meaning of the words that help us make texts more coherent and make up sentences with them within a paragraph:*

however, furthermore, while, on the one hand, on the other hand, therefore, so, thus, as, for, since (в значении *поскольку, так как*).

7. *Read the quotations below and answer the questions.*

Homer Burton Adkins (1892-1949), American organic chemist.

Basic research is like shooting an arrow into the air and, where it lands, painting a target.

1. Is math research done in the same way?

Claude Bernard (1813-78), French physiologist.

The experimenter who does not know what he is looking for will never understand what he finds.

2. Which quotation is more relevant to the humanities (sciences)?

3. Is there any contradiction between Adkins and Bernard?

4. What was the idea behind the quotations?

II. Falling Body

Isaak Asimov, Understanding Physics, First Mentor Printing, April, 1969.

Let's consider a falling body again.

An object held at some point above the ground is at rest. If it is released, it begins to fall at once. Motion is apparently created where it did not previously exist. But the word "created" is a difficult one for physicists (or for that matter philosophers) to swallow. Can anything really be created out of nothing? Or is one thing *merely* changed into a second, so the second comes into existence only at the expense of the passing into nonexistence of the first? Or perhaps one object undergoes a change (from rest to motion, for instance) because, and only because, another object undergoes an opposing change (from rest to motion in the opposite direction, for instance). In this last case, what is created is not motion but motion plus "anti-motion", and if the two together cancel out to zero, there is perhaps no true creation at all.

To straighten this matter out, let's start by trying to decide exactly what we mean by motion.

We can begin by saying that a force certainly seems to create motion. Applied to any body initially at rest, say to a hockey puck on ice, a force initiates an acceleration and sets the puck moving faster and faster. The longer the force acts, the faster the hockey puck moves. If the force is constant, then the velocity at any given time is proportional to the amount of the force multiplied by the time during which it is applied. The term impulse (I) is applied to this product of force (f) and time (t) :

$I = ft$ (Equation 6-1)

Since a force produces motion, we might expect that a given impulse (that is a given force acting over a given time) would always produce the same amount of motion. If this is so, however, then the amount of motion cannot be considered a matter of velocity alone. If the same force acts upon a second hockey puck ten times as massive as the first, it will produce a smaller acceleration and in a given time will bring about a smaller velocity than in the first case. The quantity of motion produced by an impulse must therefore involve mass as well as velocity. That this is indeed so is actually implied by Equation 6-1. By Newton's second Law we know that a force is equal to mass times acceleration ($f = ma$). We can therefore substitute ma for f in Equation 6-1 and write:

$I = mat$ (Equation 6-2)

But by Equation $v = at$, we know that for any body starting at rest the velocity (v) produced by a force is equal to the acceleration (a) multiplied by time (t), as that $at = v$. It we substitute v for at in Equation 6-2 we have:

$I = mv$ (Equation 6-3)

It is this quantity, mv, mass times velocity that is really the measure of the motion of a body. A body moving rapidly requires a greater effort to stop it than does the same body moving slowly. The increase in velocity adds to is total motion therefore. On the other hand, a massive body moving at a certain velocity requires a greater effort to stop it than does a light body moving at the same velocity. The increase in mass also adds to total motion. Consequently, the product mv has come to be called momentum (from Latin word for "motion").

Equation 6-3 means that an impulse (ft) applied to a body at rest causes that body to gain a momentum (mv) equal to the impulse. More generally, if the body is already in movement, the application of an impulse brings about a change of momentum, equal to the impulse. In brief, impulse equals change of momentum.

The units of impulse must be those of force multiplied by those of time, according to Equation 6-1, or those of mass multiplied by those of velocity according to equation 6-3. In the *mks* system, the units of force are newtons, so impulse may be measured in $newton - sec$. The units of mass are kilograms, however, and the units of velocity an meters per second, so the units of impulse (mass times velocity) are $kg - m/sec$. However, a newton has been defined as a $kg - m/sec^2$.

A newton-sec, therefore, is a $kg - m - sec/sec^2$, or a $kg - m/sec$. Thus the units of I considered as ft are the same as the units of I considered as mv. In the cgs system, it is easy to show the units of impulse are $dyne - sec$, or $gm - cm/sec$, and these are identical also.

Exercises

1. *Translate the following sentences into Russian, paying attention to the italicized words and structures:*

1) *The longer* the force acts, *the faster* the hockey puck moves.

2) *Since* a force produces motion, we *might* expect that a given impulse (that is a given force acting over a given time) *would* always *produce* the same amount of motion.

3) If the same force acts upon a second hockey puck *ten times as* massive *as* the first, it will produce a smaller acceleration and in a given time will bring about a smaller velocity than in the first case.

4) *It is this quantity* – mass times velocity – *that* is really the measure of the motion of a body.

5) The units of impulse must be *those* of force multiplied by *those* of time.

2. *Specify the italicized verbal forms in the sentences below and translate them into Russian:*

1) Let's consider a *falling* body.

2) An object *held* at some point above the ground is at rest.

3) Or perhaps one object undergoes a change because, and only because, another object undergoes an *opposing* change.

4) ... for any body *starting* at rest the velocity (v) produced by a force is equal to the acceleration (a) *multiplied* by time (t)... $at = v$.

3. *Give the English equivalents of the following Russian terms:*

Движение, сила, тело, ускорение, скорость, импульс, произведение, единица измерения.

4. *Add the missing letters to complete the words below:*

Eq...ation, rel......se, ac......leration, prev...o...sly, ph...si...ist, ap...ar...ntly, exist...nce, op...o...ite, for...e, init...ate, v...lo...ity, m...lt...pl...

5. *Remember the following word combinations and set expressions.*

1) To come into existence – возникать

2) In the opposite direction – в противоположном направлении

3) To produce / initiate acceleration – вызывать ускорение

4) To produce velocity – приводить к появлению скорости

5) At some point – в некоторой точке

6) To undergo a change – претерпевать изменения

7) From rest to motion – из состояния покоя в состояние движения

8) To produce/ create motion – порождать движение

9) To straighten this matter out – для того чтобы прояснить этот момент

10) To gain a momentum – получать импульс

11) To be in movement – находиться в движении

12) To be equal to – быть равным чему-то

13) To apply to a body – прикладывать к объекту (например, силу)

14) An increase in velocity – увеличение скорости

15) To act upon a body – действовать на тело (о силе)

16) To substitute A for B – заменять B на A

Now use them in the sentences below:

1. Увеличение скорости влияет на состояние движущегося тела. 2. По-видимому, движение возникает там, где его ранее не было. 3. Согласно второму закону Ньютона сила равна произведению массы и ускорения. 4. Один объект претерпевает изменение (например, переходит из состояния покоя в состояние

движения) только потому, что какой-то другой объект претерпевает противоположное изменение (т.е., например, переходит из состояния движения в состояние покоя и движется в противоположном направлении). 5. Для того чтобы прояснить этот вопрос, необходимо понять, что означает термин «движение». 6. Сила, приложенная к телу, приводит тело, находящееся в покое, в состояние движения.

6. *Remember the following idioms and expressions:*

1) At rest – в состоянии покоя

2) On the one hand, ... on the other hand – с одной стороны, ... с другой стороны

3) For that matter – если уж на то пошло

4) At all – совсем, полностью

5) At once – сразу

6) At the expense of smth – за счет чего-либо

7) For instance – например

8) To be proportional to smth – быть пропорциональным чему-то

9) At any given time – в любой заданный момент времени

10) Multiplied by – умноженный на

11) Over a given time – в течение данного периода времени

12) According to/ By Newton's second law – в соответствии со вторым законом Ньютона

13) To move at the same velocity – двигаться с той же скоростью

14) Per second – в секунду

Add the missing prepositions:

1. What is created is not motion but motion plus "anti-motion", and if the two together cancel out to zero, there is perhaps no true creation ... all. 2. He wasn't right. Neither was I, ... that matter. 3. If the force is constant, then the velocity ... any given time is proportional ... the amount of the force multiplied ... the time during which it is applied. 4. A force acts ... a given time. 5. ... Newton's second law we know that a force is equal to mass times acceleration $f = ma$. 6. ... the other hand, a massive body moving

... a certain velocity requires a greater effort to stop it than does a light body moving ... the same velocity. 7. The units of mass are kilograms, and the units of velocity are metres ... second.

Now translate the sentences using the above expressions:

1. Предмет, подвешенный в некоторой точке над землей, находится в состоянии покоя. 2. Если его отпустить, он сразу же начнет падать. 3. Одно явление возникает за счет того, что какое-то другое явление перестало существовать.

7. *Find the following phrasal verbs in the text and try to guess their meaning.*

1) To add to 2) to bring about 3) to cancel out

Now use them to complete the sentences below:

1. Our attempts to prove the theorem failed, and this only ... ... our difficulties. 2. What ... ... the change in the curriculum? 3. The advantages and disadvantages of this scientific approach seem to ... each other ....

8. *Explain the use of the articles in the following sentences:*

1) "Can anything really be created out of nothing? Or is one thing merely changed into *a* second, so *the* second comes into existence only at *the* expense of *the* passing into *the* non-existence of the first?"

2) "If *the* same force acts upon *a* second hockey puck ten times as massive as the first, it will produce *a* smaller acceleration and in *a* given time will bring about *a* smaller velocity than in the first case."

3) "*A* body moving rapidly requires *a* greater effort to stop it than does *the* same body moving slowly."

9. *While reading the text you might have noticed that the ideas are linked by means of the so-called connectors, or linking words. There are a number of such linking devices that help to connect ideas and convey a variety of senses.*

Thus, to compare two points in a text or indicate that they are different you may write: *likewise/ similarly/ like/ in exactly the same way....* Or you may write: *alternatively, as distinct from, as opposed to, conversely, in/by contrast, however, on the other hand, unlike, whereas, whilst/while.* If you need to give examples or more detail of a point, you may use: *e.g., for example, for instance, such as; as an example, consider...; examples include...; by way of illustration...; the following may serve as an example; this can be exemplified/illustrated as follows:...; to exemplify/illustrate this point we may consider...* or *a case in point, in particular, i.e., namely, specifically.* Other examples of connectors include: *beyond doubt/ question, undoubtedly, unquestionably, there can be no question/ doubt that..., it is undeniable that...; as a rule, by and large, generally speaking, in general, in most cases, on the whole.*

(See Macmillan English Dictionary for Advanced Learners. International Student Edition, pp. 279, 476)

Now find other linking words and phrases in the text and write a summary of the article. Remember to use linking devices to make it logical.

10. *Read the quotations and answer the questions.*

Josiah Willard Gibbs (1839-1903), American theoretical physicist and chemist.

A mathematician may say anything he pleases, but a physicist must be at least partially sane.

Georg Cantor (1845-1918), German mathematician.

The essence of mathematics lies precisely in its freedom.

1. What kind of freedom did Cantor mean?

2. Do you think Gibbs' quotation is insulting? Why, why not?

3. Try to explain what Gibbs meant.

4. Do you think there is any connection between the quotations?

III. Conservation of Momentum

Isaak Asimov, Understanding Physics, First Mentor Printing, April, 1969.

Imagine a hockey puck of mass m speeding across the ice at a velocity, v. Its momentum is mv. Imagine another hockey puck of the same mass moving at the same speed but in the opposite direction. Its velocity is therefore is $-v$ and its momentum is $-mv$. Momentum, you see, is a vector, since it involves velocity, and not only has quantity but direction. Naturally, if we have two bodies with momenta in opposite directions, we can set one momentum equal to some positive value and the other equal to some negative value.

Suppose now that the two hockey pucks are rimmed with a layer of glue powerful enough to make them instantly stick together on contact. And suppose they do make contact head-on. When that happens, they would come to an instant halt.

Has the momentum been destroyed? Not at all. The total momentum of the system was $mv + (-mv)$, or 0, before the collision and $0 + 0$, or (still) 0, after the collision. The momentum was distributed among the parts of the system differently before and after the collision, but the total momentum remained unchanged.

Suppose that instead of sticking when they collided (an inelastic collision) the two pucks bounced with perfect springiness (an elastic collision). It would then happen that each puck would reverse directions. The one with the momentum mv would now have the momentum $-mv$ and vice versa. Instead of the sum $mv + (-mv)$, we would have the sum $(-mv) + mv$. Again there would be a change in the distribution of momentum, but again the total momentum of the system would be unchanged.

If the collision were neither perfectly elastic nor completely inelastic, if the puck, bounced apart but only feebly, one puck might change from mv *to* $-0.2mv$*, while the other changed from* $-mv$ *to* $0.2mv$*.The final sum would still be zero.*

This would still hold true if the pucks met at an angle, rather than head-on, and bounced glancingly. If they met at an angle, so their velocities were not in exactly opposite directions, the two momenta would not add up to zero, even though the velocities of the two pucks were equal. Instead the total momentum of the system would be arrived at by vector addition of the two individual momenta. The two pucks would then bounce in such a way that the vector addition of the two momenta after the collision would yield the same total momentum as before. This would also be true if a moving puck struck a puck at rest a glancing blow. The puck at rest would be placed in motion, and the originally moving puck would change its direction; *however, the two final momenta would add up to the original.*

Matters would remain essentially unchanged even it the two pucks were of different masses. Suppose one puck was moving to the right at a given speed and had a momentum of mv, while another, three times as massive, was moving at the same speed to the left and had, therefore, a speed of $-3mv$. If the two stuck together after a head-on collision, the combined pucks (with a total mass of 4m) would continue moving to the left – the direction in which the more massive puck had been moving – but at half the original velocity $(-v/2)$. The original momentum of the system was $mv + (-3mv)$, or $-2mv$. The final momentum of the system was $(4m)(-v/2)$, or $-2mv$. Again, the total momentum of the system would be unchanged.

And what if momentum is seemingly created? Let us consider a bullet initially at rest – and with a momentum, therefore, of 0

– which is suddenly fired out of a gun and moves to the right at high velocity. It now has considerable momentum (mv). However, the bullet is only part of the system. The remainder of the system, the gun, must gain $-mv$ by moving in the opposite direction. If the gun has (n) times the mass of the bullet, it must move in the opposite direction with l/n times the velocity of the speeding bullet. The momentum of the gun (minus the bullet) would then be $(nm)(-v/n)$ or $-mv$. *(If the gun were suspended freely when it was fired, its backward jerk would be clearly visible.* When fired in the usual manner its backward motion is felt as recoil.) The total momentum of gun plus bullet was therefore 0 before the gun was fired and 0 after it was fired, though here the distribution of momentum among the parts of the system varied quite a bit before and after firing.

In short, all the experiments we can make will bring us to the conclusion that:

The total momentum of an isolated system of bodies remains constant.

This is called the law of conservation of momentum. (Some thing that is "conserved" is protected, guarded, or kept safe from loss.)

Of course, it is impossible to prove a generalization by merely enumerating isolated instances. No matter how often you experiment and find that momentum is conserved, you cannot state with certainty that it will always be conserved. At best, one can only say, as experiment after experiment follows the law and as no experiment is found to contradict it, that the law is increasingly probable. *It would be far better if one could show the generalization to be a consequence of another generalisation that is already accepted.*

For instance, suppose two bodies of any masses and moving at any velocities collide at any angle with any degree of elasticity.

At the moment of collision, one body exerts a force (f) on the second. By Newton's third law, the second body exerts an equal and opposite force ($-f$) on the first. The force is exerted only while the two bodies remain in contact. *The time (t) of contact is obviously the same for both bodies, for when the first is no longer in contact* *with the second; the second is no longer in contact with the first.* This means that the impulse of the first body on the second is ft, and that of the second on the first is $-ft$.

The impulse of the first body on the second imparts a change in momentum mv to the second body. But the impulse of the second body on the first, being exactly equal in quantity but opposite in sign, must impart a change in momentum $-mv$ to the first. The changes in momentum may be large or small depending on the size of the impulse, the angle of collision, and the elasticity of the material; however, whatever the change in momentum of one, the change in the other is equal in size and opposition in direction. The total momentum of the system must remain the same.

Thus, the law of conservation of momentum can be derived from Newton's third law of motion. In actual fact, however, it was not, for the law of conservation of momentum was first enunciated by an English mathematician, John Wallis (1616-1703), in 1671, a dozen years before Newton published his laws of motion. *One could, indeed, work it the other way, and derive the third law of motion from the law of conservation of momentum.*

At this point you might feel that if the physicist proves the conservation of momentum from the third law of motion, and then proves the third law of motion from the conservation of momentum, he is actually arguing in a circle and not proving anything at all. *He would be if that were what he is doing, but he is not.*

It is not so much a matter of "proving" as of making an assumption and demonstrating a consequence. One can begin by assuming the third law of motion and then showing that the law of conservation of momentum is a consequence of it. Or one can begin by assuming the law of conservation of momentum and showing that the third law is the consequence of that.

The direction in which you move is merely a matter of convenience. In either case, no proof is involved and no necessary truth. The whole structure rests on the fact that no one in nearly three centuries has been able to produce a clear-cut demonstration that a system exists, or can be prepared, in which either the third law of motion or the law of conservation of momentum is not obeyed. *Such a demonstration may be made tomorrow, and the* *foundations of physics may have to be modified as a consequence*; but by now it seems very unlikely that this will happen.

And yet it may be that with a little thought we might think of cases where the law is not obeyed. For instance, suppose a billiard ball hits the rim of the billiard table squarely and rebounds along its own line of approach. Its velocity v becomes $-v$ after the rebound, and since its mass remains unchanged, its original momentum mv has become $-mv$. Isn't that a clear change in momentum?

Yes, it is, but the billiard ball does not represent the entire system. The entire system includes the billiard table that exerted the impulse that altered the billiard ball's momentum. Indeed, since the billiard table is fixed to the ground by frictional forces too large for the impact of the billiard ball to overcome, it includes the entire planet. The momentum of the earth changes just enough to compensate for the change in the momentum of the billiard ball. However, the mass of the earth is vastly larger than that of the billiard ball, and its change in velocity is therefore correspondingly smaller – far too small to detect by any means known to man.

Yet one might assume that if enough billiard balls going in the same direction were bumped into enough billiard tables, at long, long last, the motion of the earth would be perceptibly changed. Not at all! Each rebounding billiard ball must strike the opposite rim of the table, or your hand, or some obstacle. Even if it comes to a slow halt through friction, that will be like striking the cloth of the table little by little. No matter how the billiard ball moves it will have distributed its changes in momentum equally in both directions before it comes to a halt, if only itself and the earth are involved.

A more general way of putting it is that the distribution of momentum among the earth and all the movable objects on or near its surface may vary from time to time, but the total momentum, and therefore the net velocity of the earth plus all those movable objects (assuming the total mass to remain constant), must remain the same. No amount or kind of interaction among the components of a system can alter the total momentum of that system.

And now the solution to the problem of the falling body with which I opened the chapter is at hand. *As the body falls it* *gains momentum* (mv)*, this momentum increasing as the velocity increases.* The system, however, does not consist of the falling body alone. The gravitational force that brings about the motion involves both the body and the earth. Consequently, the earth must gain momentum ($-mv$) by rising to meet the body. Because of the earth's huge mass, its upward acceleration is vanishingly small and can be ignored in any practical calculation. Nevertheless, the principle remains. Motion is not created out of nothing when a body falls. Rather, both the motion of the body and the anti-motion of the earth are produced, and the two cancel each other out. The total momentum of earth and falling body, with respect to each other, is zero before the body starts falling, is zero after it completes its fall, and is zero at every instant during its fall.

Exercises

1. *In the text, translate the sentences italicized.*

2. *Insert prepositions:*

... a velocity, the two momenta add ... ... zero, ... rest, to move ... the right ... a given speed, to exert a force ... a body, ... Newton's third law, to depend ... the size, to derive ... Newton's third law, to detect ... any means known to man, to vary ... time .. time, solution ... a problem.

3. *Give the English equivalents of:*

1) предположим, что
2) полный импульс
3) неупругое столкновение
4) и наоборот
5) это выполняется, если
6) в три раза тяжелее
7) со скоростью в два раза меньшей первоначальной
8) довольно сильно отличаться (до и после чего-либо)
9) вкратце, одним словом
10) определенно утверждать
11) в лучшем случае

12) в любом из двух случаев
13) подчиняться закону
14) вызывать движение
15) ускорение пренебрежимо мало

4. *Using your English-English dictionary, check the pronunciation and stress of the words below:*

distribute, vice versa, yield, merely, consequence, sign, either.

5. *Make up sentences of your own with the following words:*

to suppose, to be distributed, to collide, to hold true, at rest, to have n times the mass of smth, to carry out an experiment, to state with certainty, to contradict, by some law, to derive, to overcome, to interact.

6. *Answer the following questions:*

1) Do you think the author's method to arrive at the law of conservation of momentum is satisfactory?
2) Could you suggest another method?
3) Could you think of other examples illustrating the law in question?
4) Could you give an example (other than in the text) that seemingly disobeys the law of conservation of momentum?

7. *Read the quotation below and answer the questions.*

Karl Raimund Popper (1902-94), Austrian-born British philosopher of science.

What really makes science grow is new ideas, including false ideas.

1. How can in principle false ideas help science develop?

2. Give examples from physics, chemistry and other sciences that support Popper's ideas.

3. Is it possible to apply the quotation to mathematics?

IV. Mechanical Energy

Isaak Asimov, Understanding Physics, First Mentor Printing, April, 1969.

It is neat and pleasant to see that the work put into one end of a lever is equal to the work coming out of the other end and *we might fairly suspect that there was such a thing as "conservation of work".*

Unfortunately, such a possible conservation law runs into a snag almost at once. After all, where did the work come from that was put into the lever? If one end of the lever was manipulated by a human being who was using the lever to lift a weight, the work came from that done by the moving human arm.

And where does the work of the moving arm come from? *A man sitting quietly can suddenly move his arm and do work where no work had previously seemed to exist. This runs counter to the notion of conservation in which the phenomenon being conserved can be neither created nor destroyed.*

If one is anxious to set up a conservation law involving work, therefore, one might suppose that work, or something equivalent to work, could be stored in the human body (and perhaps in other objects) and that this work-store could be called upon at need and converted into visible, palpable work.

At first blush such a work-store might have seemed to be particularly associated with life, since living things seemed filled with this capacity to do work, whereas dead things, for the most *part, lay quiescent and did not work.* The German philosopher and scientist Gottfried Wilhelm Leibnitz (1646-1716), who was the first to get a clear notion of work in the physicist's sense, chose to call this work-store vis viva (Latin for "living force"). However, it is clearly wrong to suppose that work is stored only in living things; as a matter of fact, the wind can drive ships and running water can turn millstones, and in both cases force is being exerted through a distance. Work, then, was obviously stored in inanimate objects as well as in animate ones. In 1807, the English physician Thomas Young (1773- 1829) proposed the term energy for this work-store. This is from Greek words meaning "work-within" and is a purely neutral term that can apply to any object, living or dead.

This term gradually became popular and is now applied to any phenomenon capable of conversion into work. There are many varieties of such phenomena and therefore many forms of energy.

The first form of energy to be clearly recognized as such, perhaps, was that of motion itself. Work involved motion (since an object had to be moved through a distance), so it was not surprising that motion could do work. It was moving air, or wind, that drove a ship, and not still air; moving water that could turn a millstone, and not still water. It was not air or water that contained energy then, but the motion of the air or water. *In fact, anything that moved contained energy, for if the moving object, whatever it was, collided with another, it could transfer its momentum to that second object and set its mass into motion; it would thus be doing work upon it, for a mass would have moved through a distance under the urging of a force.*

The energy associated with motion is called kinetic energy, a term introduced by the English physicist Lord Kelvin (1824-1907) in 1856. The word "kinetic" is from a Greek word meaning "motion."

Exactly how much kinetic energy is contained in a body moving at a certain velocity, v? To determine this, let us assume that in the end we are going to discover that there exists a conservation law for work in all its forms – stored and otherwise. In that case, we can be reasonably confident that if we find out how much work it takes to get a body moving at a certain velocity, v, then that automatically will be the amount of work it can do on some other object through its motion at that velocity. In short, that would be its kinetic energy.

To get a body moving in the first place takes a force, and that force, by Newton's second law, is equal to the mass of the moving body multiplied by its acceleration:

$f = ma$ (Equation 7-1)

The body will travel for a certain distance, d, before the acceleration brings it up to the velocity, v, which we are inquiring into. The work done on the body to get it to that velocity is the force multiplied by the distance. Expressing the force as ma we have:

$w = mad$ (Equation 7-2)

Now much earlier in the book, in discussing Galileo's experiments with falling bodies, we showed that $v = at$ - that velocity, in other words, is the product of acceleration and time. This is easily rearranged to: $t = v/a$.

We also pointed out in discussing Galileo's experiments that where there is uniform acceleration,

$d = 1/2at^2$ where d is the distance covered by the moving body.

If, in place of t in the relationship just given, the quantity v/a is substituted, we have:

$d = 1/2a(v/a)^2 = 1/2(v^2)/a$ (Equation 7-3)

Let us now substitute this value for d in Equation 7-2 which becomes:

$w = 1/2(mav^2)/a = 1/2mv^2$ (Equation 7-4)

This is the work that must be done upon a body of mass m to get it to move at a velocity v, and it is therefore the kinetic energy contained by the body of that mass and with that velocity. It we symbolize kinetic energy as $e(k)$ we can write:

$e_k = \frac{1}{2}mv^2$ (Equation 7-5)

I have already pointed out that work has the units of mass multiplied by those of velocity squared and, as is clear from Equation 7-5, so has kinetic energy. Therefore, kinetic energy can be measured in joules or ergs, as can work. Indeed, all forms of energy can be measured in these units.

We might now imagine that we can set up a conservation law in which kinetic energy can be converted into work and vice versa, but in which the sum of kinetic energy and work in any isolated system must remain constant. Such a conservation law will not, however, hold water, as can easily be demonstrated.

An object thrown up into the air has a certain velocity and therefore a certain kinetic energy as it leaves the hand (or the catapult or the cannon). As it climbs upward, its velocity decreases because of the acceleration imposed upon it by the earth's gravitational field. Kinetic energy is therefore constantly disappearing and, eventually, when the ball reaches maximum height and comes to a halt, its kinetic energy is zero, and has therefore entirely disappeared.

One might suppose that the kinetic energy has disappeared because work has been done on the atmosphere, and that therefore kinetic energy has been converted into work. *However, this is not an adequate explanation of events, for the same thing would happen in a vacuum.*

One might next suppose that the kinetic energy had disappeared completely and beyond redemption, without the appearance of work, and that no conservation law involving work, and energy was therefore possible. However, after an object has reached maximum height and its kinetic velocity has been reduced to zero, it begins to fall again, still under the acceleration of gravitational force. It falls faster and faster, gaining more and more kinetic energy, and when it hits the ground (neglecting air resistance) it possesses all the kinetic energy with which it started.

Rather than lose our chance at a conservation law, it seems reasonable to assume that energy is not truly lost as an object rises upward, but that it is merely stored in some form other than kinetic energy. Work must be done on an object to lift it to a particular height against the pull of gravity, even if once it has reached that height it is not moving. This work must be stored in the form of an energy that it contains by virtue of its position with respect to the gravitational field.

Kinetic energy is thus little by little converted into "energy of position" as the object rises. At maximum height, all the kinetic energy has become energy of position. As the object falls once more, the energy of position is converted back into kinetic energy. Since the energy of position has the potentiality of kinetic energy, the Scottish engineer William J. M. Rankine (1820-1872) suggested, in 1853, that it be termed *potential energy*, and this suggestion was eventually adopted.

To lift a body a certain distance (d) upward, a force equal to its weight must be exerted through that distance. The force exerted by a weight is equal to mg. where m is mass and g the acceleration due to gravity (see Equation 5-1). If we let potential energy be symbolized as $e(p)$ we have:

$e_p = mgd$ (Equation 7-6)

If all the kinetic energy of a body is converted into potential energy, then the original e_k is converted into an equivalent $e(p)$ or combining Equations 7-5 and 7-6:

$$1/2mv^2 = mgd$$

or simplifying, and assuming g to be constant.

$v^2 = 2gd = 19.6d$ (Equation 7-7)

From this relationship one can calculate (neglecting air resistance) the height to which an object will rise if the initial velocity with which it is propelled upward is known. The same relationship can be obtained from the equations arising out of Galileo's experiments with falling objects.

Kinetic energy and potential energy are the types of energy made use of by machines built up out of levers, inclined planes and wheels, and the two forms may therefore be lumped together as mechanical energy. *As long ago as the time of Leibnitz it was recognized that there was a sort of "conservation of mechanical energy,"* and that (if such extraneous factors as friction and air resistance were neglected) mechanical energy could be visualized as bouncing back and forth between the kinetic form and the potential form, or between either and work, but not (taken in all three forms) as appearing from nowhere or disappearing into nowhere.

Exercises

1. *In the text, translate the sentences italicized.*

2. *Insert prepositions where necessary:*

to run ... a snag, to run counter ... a notion, to set ... a law, equivalent ... smth, to convert smth ... smth, ... the most part, ... a matter ... fact, capable ... smth, to set a mass ... motion, to move ... a velocity, to be equal ... smth, to rearrange ... some form, to substitute smth ... smth, to do work ... some object, velocity reduces ... zero, ... virtue of smth, position ... respect ... smth, ... the gravitational field, ... maximum height, acceleration due ... gravity, to multiply smth ... smth.

3. *Give the English equivalents of:*

1) производить работу над чем-либо
2) установить (вывести) закон
3) при необходимости
4) по большей части
5) фактически, на самом деле
6) физическое явление
7) неподвижный воздух
8) привести тело в движение
9) количество работы
10) произведение ускорения и времени
11) расстояние, пройденное телом
12) как видно из уравнения
13) измерять в этих единицах
14) полностью исчезать (об энергии)
15) мало-помалу, постепенно
16) начальная энергия
17) начальная скорость
18) наклонная плоскость
19) пренебрегать сопротивлением

4. *Using your English-English dictionary, check the pronunciation and stress of the words below:*

lever, neither ... nor, physicist, inanimate (adj), joule, vice versa, climb, merely, virtue, assume, assumption, kinetic.

5. *Describe the contributions of Leibnitz, Young, Kelvin and Rankine to the development of the "energy" concept in 6-9 sentences.*

6. *Read the quotations below and answer the questions.*

Anonymous

Fiction tends to become 'fact' simply by serial passage via the printed page.

Reijer Hooykaas (1906-94), Dutch historian of science.

The history of science shows so many examples of the 'irrational' notions and theories of today becoming the 'rational' notions and theories of tomorrow, that it seems largely a matter of being accustomed to them whether they are considered rational or not, natural or not.

1. Could you give any examples of such notions and theories (not necessarily mathematical)?

2. Do you think this quotation may be applied to mathematics? To the sciences? To the humanities?

3. Could you give any examples?

V. Conservation of Energy

Isaak Asimov, Understanding Physics, First Mentor Printing, April, 1969.

Unfortunately, the "law of conservation of mechanical energy," however neat it might seem under certain limited circumstances, has its imperfections, and these at once throw it out of court as a true conservation law.

An object hurled into the air with a certain kinetic energy, returns to the ground without quite the original kinetic energy. A small quantity has been lost through air resistance. *Again, if an elastic object is dropped from a given height, it should (if mechanical energy is to be truly conserved) bounce and return to exactly its original height.* This it does not do. It always returns to somewhat less than the original height, and if allowed to drop again and bounce and drop again and bounce, it will reach lower and lower heights until it no longer bounces at all. Here it is not only the air resistance that interferes but also the imperfect elasticity of the body itself. Indeed, if a lump of soft clay is dropped, its potential energy is converted to kinetic energy, but at the moment it strikes the ground with a non-bouncing splat that kinetic energy is gone – and without any reformation of potential energy. To all appearances, mechanical energy disappears in these cases.

One might argue that these losses of mechanical energy are due to imperfections in the environment. *If only a frictionless system were imagined in a perfect vacuum, if all objects were completely elastic, then mechanical energy would be conserved.*

However, such an argument is quite useless, for in a true conservation law the imperfections of the real world do not affect the law's validity. Momentum is conserved, for instance regardless of friction, air resistance, imperfect elasticity or any other departure from the ideal.

If we still want to seek a conservation law that will involve work we must make up our minds that for every loss of mechanical energy there must be a gain of something else. That something else is not difficult to find. Friction, one of the most prominent imperfections of the environment, will give rise to heat, and if the friction is considerable, the heat developed is likewise considerable. (The temperature of a match-head can be brought to the ignition point in a second by rubbing it against a rough surface.

Conversely, heat is quite capable of being turned into mechanical energy. The heat of the sun raises countless tons of water vapor kilometres high into the air, so that all the mechanical energy of falling water (where as rain cataracts or quietly flowing rivers) must stem from the sun's heat. Furthermore, the eighteenth century saw man deliberately convert heat into mechanical energy by means of a device destined to reshape the world. Heat was used to change water into steam in a confined chamber, and this steam was then used to turn wheels and drive pistons. (Such a device is, of course, a steam engine.)

It seemed clear, therefore, that one must add the phenomenon of heat to that of work, kinetic energy and potential energy, in working out a true conservation law. Heat, in short, would have to be considered another form of energy.

But if that is so, then any other phenomenon that could give rise to heat would also have to be considered a form of energy. An electric current can heat a wire and a magnet can give rise to an electric current, so both electricity and magnetism are forms of energy. Light and sound are also forms of energy, and so on.

If the conservation law is to encompass work and all forms of energy (not mechanical energy alone), then it had to be shown that one form of energy could be converted into another quantitatively. In other words, in such energy-conversions all energy must he accounted for; no energy must be completely lost in the process, no energy created.

This point was tested thoroughly over a period of years in the 1840's by an English brewer named James Prescott Joule (1818-1889), whose hobby was physics. He measured the heat produced by an electric current, that produced by the friction of water against glass, by the kinetic energy of turning paddle wheels in water, by the work involved in compressing gas, and so on. In doing so, he found that a fixed amount of one kind of energy was converted into a fixed amount of another kind of energy, and that if energy in all its varieties was considered; no energy was either lost or created. *It is in his honor that the unit of work and energy in the mks system is named the "joule."*

In a more restricted sense, one can consider that Joule proved that a certain amount of work always produced a certain amount of heat. Now the common British unit of work is the "foot-pound" — that is, the work required to raise one pound of mass through a height of one foot against the pull of gravity. The common British unit of heat is the "British thermal unit" (commonly abbreviated "Btu") which is the amount of heat required to raise the temperature of one pound of water by 1^o Fahrenheit. Joule and his successors determined that 778 foot-pounds are equivalent to 1 Btu, and this is called the *mechanical equivalent of heat.*

It is preferable to express this mechanical equivalent of heat in the metric system of units. A foot-pound is equal to 1.356 joules, so 778 foot-pounds equal 1055 joules. Furthermore, the most common unit of heat in physics is the calorie, which is the amount of heat required to raise the temperature of one gram of water by 10 Centigrade. One Btu is equal to 252 calories. Therefore, Joule's mechanical equivalent of heat can be expressed as 1055 joules equal 252 calories, or 4.18 joules = 1 calorie.

Once this much was clear, it was a natural move to suppose that the law of conservation of mechanical energy should be converted into a law of conservation of energy, in the broadest sense of the word – including under "energy," work, mechanical energy, heat, and everything else that could be converted into heat. Joule saw this, and even before his experiments were far advanced, *a German physicist named Julius Robert von Mayer (1814-1878) maintained it to be true.* However, the law was first explicitly stated in form clear enough and emphatic enough to win acceptance by the scientific community in 1847 by the German physicist and biologist Hermann von Helmholtz (1821-1894), and *it is he who is generally considered the discoverer of the law.*

The law of conservation of energy is probably the most fundamental of all the generalisations made by scientists and the one they would be most reluctant to discard. As far as we can tell it holds through all the departures of the real universe from the ideal models set up by scientists; it holds for living systems as well as nonliving ones; and for the tiny world of the subatomic realm as well as for the cosmic world of the galaxies. At least twice in the last century phenomena were discovered which seemed to violate the law, but both times physicists were able to save matter by broadening the interpretation of energy. In 1905, Albert Einstein showed that mass itself was a form of energy; and in 1931, the Austrian physicist Wolfgang Pauli (1900-1958) advanced the concept of a new kind of subatomic particle, the neutrino, to account for apparent departures from the law of conservation of energy.

Nor was this merely a matter of saving appearances or of patching up a law that was springing leaks. Each broadening of the concept of conservation of energy fit neatly into the expanding structure of twentieth-century science and helped explain a host of phenomena; *it also helped predict (accurately) another host of phenomena that could not have been explained or predicted otherwise.* The nuclear bomb, for instance, is a phenomenon that can only be explained by the Einsteinian concept that mass is a form of energy.

Exercises

1. *In the text, translate the sentences italicized.*

2. *Insert prepositions where necessary:*

... certain circumstances, regardless ... friction, departure ... the ideal, to give rise ... heat, to rub ... a rough surface, ... other words, friction of water ... glass, ... a more restricted sense, to be equal ... smth, to equal ... smth, to raise the temperature ... 10 degrees, ... least, to account ... departures from the law.

3. *Give the English equivalents of:*

1) при определенных обстоятельствах
2) упругий объект
3) неабсолютная упругость
4) кинетическая энергия исчезает
5) по всей видимости
6) можно утверждать, что
7) трение значительно
8) образовавшееся тепло
9) тереть о поверхность
10) преобразовывать тепло в механическую энергию посредством (с помощью) машины (устройства)
11) электрический ток
12) другими словами
13) сжимать газ
14) увеличить температуру газа на 10 градусов
15) явно, четко формулировать
16) довольно понятный
17) делать обобщения
18) насколько мы знаем
19) противоречить закону
20) точно предсказывать
21) последовательность действий

4. *Using your English-English dictionary, check the pronunciation and stress of the words below:*

circumstance, validity, ignition, surface, ton, encompass, thoroughly, tiny, accurate, bomb.

5. *Make sure you pronounce the following surnames correctly:*

Joule, Fahrenheit, Mayer, Helmholtz, Einstein, Pauli.

6. *Write a summary of the text in 4-5 sentences.*

7. *Read the quotations below. Answer the questions.*

Patrick Maynard Stuart Blackett (Lord Blackett) (1897-1974), British physicist.

A first-rate laboratory is one in which mediocre scientists can produce outstanding work.

Ernest Rutherford (Baron Rutherford of Nelson) (1871-1937), New Zealand-born British physicist.

We haven't the money, so we've got to think.

1. Do you think the scentists spoke about the same?

2. Is it possible nowadays to develop science without relying on sophisticated equipment, etc.?

3. In which sciences is it more likely?

VI. Coriolis Forces

I.

Many interesting phenomena occurring on the Earth are explained by the action of Coriolis forces. The Earth is a sphere and this makes the effects of Coriolis forces more complicated. These forces will not only influence motion along the Earth's surface, but also the falling of bodies to the Earth.

Does a body fall exactly along a vertical? Not quite. Only at a pole does a body fall exactly along a vertical. Here the direction of the motion and the Earth's axis of rotation coincide, so there is no Coriolis force. The situation is different at the equator; here the direction of the motion forms right angles with the Earth's axis. *If looked upon from the North Pole, the Earth's rotation will appear to be counterclockwise.* Hence, a freely falling body should be deflected to the right of its path, i.e. to the East. The magnitude of this eastward deflection, the greatest at the equator, decreases to zero as the poles are approached.

Let us compute the magnitude of the deflection at the equator. Since a freely falling body moves with a uniform acceleration, the Coriolis force increases as the Earth is approached. *We shall therefore restrict ourselves to an approximate computation. If the body falls from a height, say, of 80 m, then its fall will last about 4 sec, according to the formula* $t = \sqrt{2h/g}$. The average speed for the fall will be equal to 20 m/sec.

This is the speed that we shall substitute in our formula,

$4\pi nv$, for the Coriolis acceleration. Let us convert the value $n = 1$ revolution in 24 hours to the number of revolutions per second; 24x3600 seconds are contained in 24 hours, so n is equal to 1/86 400 rev/sec; consequently, the acceleration created by the Coriolis force is equal to $\pi/1080$ m/sec^2. The distance covered during 4 sec with such an acceleration is equal to $(1/2)(\pi/1080) * 42 = 2.3$ cm. This is precisely the magnitude of the eastward deflection in our example. An exact computation, taking into account the non-uniformity of the fall, yields a somewhat different number - 3.1 cm.

While the deflection of a freely falling body is maximal at the equator and equal to zero at the poles, we shall see the opposite picture in the case of the deflection of a body, moving in a horizontal plane, under the action of a Coriolis force.

A body moving along a horizontal site on the North or South pole will be deflected to the right of its motion's direction by the Coriolis force at the North Pole, and to the left at the South. *Using the same formula for the Coriolis acceleration, the reader can calculate without difficulty that a bullet fired from a rifle with an initial speed of 500 m/sec will be deflected from the target by 3.5 cm in a horizontal plane during one second (i.e. while it travels 500 m).*

But why should the deflection in a horizontal plane at the equator be equal to zero? Without rigorous proofs, it is clear that this should be the case. At the North Pole a body is deflected to the right of the motion's path, at the South, to the left; hence, half-way between the poles, i.e. at the equator, the deflection will be equal to zero.

II.

Let us recall the experiment with Foucault's pendulum. A pendulum oscillating at a pole preserves the plane of its oscillations. The Earth, rotating, moves away from under the pendulum. This is how a stellar observer explains Foucault's experiments. But an observer rotating together with the Earth explains this experiment by means of Coriolis force. As a matter of fact, a Coriolis force is directed perpendicularly to the Earth's axis and perpendicularly to the direction of the pendulum's motion; in the other words, the force is perpendicular to the plane of the plane of the pendulum's oscillation and will continually turn this plane. The Earth completes one quarter of a rotation during one and a half periods of the pendulum's oscillation. The Foucault pendulum turns much more slowly. At a pole, the pendulum's plane of oscillation will turn through $\frac{1}{4}$ of a degree during one minute. At the North Pole the plane will be turned to the right of the pendulum's path, and at the South, to the left.

The Coriolis effect will be somewhat less at Central European latitudes than at the equator. A bullet in the example we have just given will be deflected not by 3.5 cm, but by 2.5 cm. The Foucault pendulum will be turned by about 1/6 of a degree during one minute.

Must a gunner take the Coriolis force into account? Big Bertha, used by the Germans to shell Paris during World War I, was situated 110 km from the target. The Coriolis deflection is as much as 1600 m in such a case. This is no longer a small quantity.

If a flying projectile is sent very far without taking the Coriolis force into account, it will be deflected significantly from its course. This effect is large, not because this force is large (for a ten-ton projectile having a speed of 1000 km/hr, the Coriolis force will be about 25 kgf), but because it is exerted continually for a long period of time.

Of course, the wind's influence on a rocket projectile may be no less significant. Flight corrections made by a pilot depend on the action of the wind, the Coriolis effect and imperfections in the airplane or flying bomb.

III.

Pressure fluctuations caused by the weather are very irregular in character. At one time people thought that pressure alone determines the weather. Therefore, the following inscriptions have been placed on barometers up to the present day: clear, dry, rain, storm. One even finds the inscription “earth-quake”.

Changes in pressure really do play a big role in changing the weather. But this role is not decisive. Average or normal pressure at sea level is equal to 1013 millibars. Pressure fluctuations are comparatively small. The pressure rarely falls below 935-940 millibars or rises to 1055-1060.

The lowest pressure was registered on August 18, 1927, in the South China Sea - 885 millibars. The highest - about 1080 millibars - was registered on January 23, 1900, at the Barnaul station in Siberia (all figures are taken with respect to sea level).

Maps are used by meteorologists to analyze changes in the weather. The lines drawn on the maps are called isobars. The pressure is the same along each such line (its value is indicated). The regions of the lowest and highest pressures are the pressure “peaks” and “pockets”.

The directions and strengths of winds are related to the distribution of atmospheric pressure. Pressures are not identical at different places on the Earth’s surface, and a higher pressure “squeezes” air into places with a lower pressure. *It would seem that a wind should blow in a direction perpendicular to the isobars, i.e. where the pressure is falling most rapidly.* However, wind maps show otherwise. Coriolis forces interfere in the matter of air pressure and contribute their corrections, which are very significant.

As we know, a Coriolis force, directed to the right of the motion, acts on any body moving in the Northern Hemisphere. This also pertains to air particles. Squeezed out of places of higher pressure and into places where the pressure is lower, the particle should move across the isobar, but the Coriolis force deflects it to the right, and so the direction of the wind forms an angle of about 45^o with the direction of the isobar.

A strikingly large effect for such a small force! This is explained by the fact that the obstacles to the action of the Coriolis force - the friction between layers of air - are also very insignificant.

The influence of Coriolis forces on the direction of winds at pressure “peaks” and “pockets” is even more interesting. Owing to the action of Coriolis forces, the air leaving a pressure "peak" does not flow in all directions along radii, but moves along curved lines - spirals. These spiral air streams twist in one and the same direction and create a circular whirlwind, displacing air masses clockwise, in a high-pressure area.

The same thing also happens in a low-pressure area. In the absence of Coriolis forces, the air would flow towards this area uniformly along all radii. However, along the way air masses are deflected to the right. In this case, as is clear from the map, a circular whirlwind is formed, moving the air counter-clockwise.

Winds in low-pressure areas are called cyclones; winds in high-pressure areas are called anticyclones.

You shouldn't think that every cyclone implies a hurricane or a storm. The passing of cyclones or anticyclones through the city where we live is an ordinary phenomenon, related, it is true, more often than not to a change in weather. In many cases, the approach of a cyclone means the coming of bad weather, while the approach of an anticyclone, the coming of good weather.

IV.

Leonard Euler was born in Basel and in 1720 entered the University of Basel. The young student's mathematical talents were soon noticed and John Bernoulli, in addition to his usual lectures, gave him private lessons weekly. At 16, Euler obtained his master's degree.

The Russian Academy of Sciences was opened in 1725 at St. Petersburg. In the summer of 1727 Euler moved to St. Petersburg and free from any other duties, he was able to put all his energy into mathematical research. He became a member of the Academy and in 1733 took the place as the head of the department of mathematics. During the time he was at the Russian Academy Euler wrote his famous book on mechanics and in it, instead of implying the geometrical methods used by Newton and his pupils, Euler introduced analytical methods. He showed how the differential equations of motion of a particle can be derived and how the motion of the body can be found integrating these differential equations. This method simplified the solution of problems, and the book had a great influence on subsequent developments in mechanics.

About that time Euler became interested in elastic curves and the latter drew Euler's attention to the problem of the lateral vibration of elastic bars and to the investigation of the corresponding differential equation.

Exercises

1. Translate the italicized parts of the text. Pay attention to the grammatical constructions.

2. Give the Russian equivalents of the following expressions:

an interesting phenomenon; to be deflected to; i.e.; let us compute; the Coriolis force; an exact computation; under the action; a rigorous proof; let us recall; as a matter of fact; pendulum's motion; in other words; pendulum's oscillation; to turn about; to take into account; a rocket projectile; to depend on; up to the present day; to squeeze out; an obstacle to an action; to owe to; in this case; in addition to; lateral vibration.

3. Translate the following passage into English:

Сила Кориолиса – это сила, действующая на частицы или предметы, вызванная вращением Земли. В результате воздействия этой силы движущиеся объекты, океанские и атмосферные течения отклоняются в правую сторону в северном полушарии и в левую сторону - в южном. Это явление сильно влияет на погодные условия в масштабах всего мира. Примером действия этого эффекта является воронкообразное движение воды, выливающейся сквозь отверстие, или воздушного вихря. Название дано по имени французского математика Гаспара Гюстава де Кориолиса (Gaspard-Gustave de Coriolis, 1792 -1843).

4. Translate the sentences from Russian into English using the word combinations from the text:

1. Сила инерции Кориолиса часто являлась скрытой причиной многих необычных явлений, наблюдаемых человеком.

2. Одним из проявлений воздействия силы Кориолиса на находящееся в свободном падении тело является отклонение его траектории от строго вертикальной.

3. Эффект отклонения падающего тела от вертикали в максимальной степени проявляется на экваторе.

4. Для наблюдателя, находящегося на Северном полюсе, Земля, вращается против часовой стрелки, а на Южном полюсе – по часовой.

5. Если на экваторе бросить тело вниз с высоты 80 метров, оно отклонится от вертикали примерно на 3 см из-за действия силы Кориолиса.

6. Под воздействием силы Кориолиса происходит изменение траектории движения тела, движущегося горизонтально.

7. Если тело движется горизонтально на Северном полюсе, то из-за силы Кориолиса оно отклонится от горизонтали вправо, а на Южном – влево.

8. Профессиональный стрелок из оружия должен знать, что выпущенный им снаряд отклонится от горизонтали примерно на 3,5 см, пройдя расстояние 500 метров.

9. Одной из причин отклонения маятника Фуко от вертикальной плоскости его колебаний является сила инерции Кориолиса, направленная перпендикулярно к направлению земной оси.

10. Родившийся в Германии выпускник университета Базеля Леонард Эйлер был приглашен в Российскую академию наук в начале восемнадцатого века.

11. В 1733 году действительный член Российской академии наук Леонард Эйлер возглавил ее отделение математики.

12. Свою самую известную книгу по механике, в которой были развиты идеи Исаака Ньютона, Леонард Эйлер написал во время своей работы в Российской академии наук.

13. Леонард Эйлер первым описал движение тел с помощью

дифференциальных уравнений и дал методы их решения, позволяющие определить траекторию движения этих тел.

14. Сила Кориолиса влияет также на распределение давления воздушных масс на территории Земли, что учитывается метеорологами, определяющими давление в разных точках земной поверхности.

15. За все время проводимых наблюдений была отмечена максимальная величина давления воздуха на земной поверхности, равная 1080 миллибар, а минимальная – 885 миллибар.

5. Read the text again and write a summary (approximately 250 words). Remember to use linking devices to make your summary coherent.

6. Read the quotations below and answer the questions.

Alfonso X (1221-84), Spanish king and astronomer.

If the Lord Almighty had consulted me before embarking upon his creation, I should have recommended something simpler.

Albert Einstein (1879-1955), German-born American physicist, and **Leopold Infeld** (1898-1968), Polish physicist.

Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.

1. Why do you think the ideas about the world are so different?

2. Could it be caused by the ages the scientists lived in?

3. Give some examples justifying both statements.

Carl Wilhelm Wolfgang Ostwald (1883-1943), Latvian chemist.

$H = W - R$

Happiness is equal to work minus resistance.

4. Do you think there should be more parameters in the formula?

5. Give your variant of a modified formula.

VII. Numbers and Measures in the Earliest Written Records

Joran Friberg, Scientific American, Vol. 250, 1986, pp. 110-118.

As early as the end of the fourth millennium B.C. proto-Sumerian and proto-Elamite scribes had well-developed systems of numbers and measures. They included precursors of our own decimal system.

Among the world's earliest written records are inscriptions on clay tablets unearthed in Iraq and Iran, in particular at the sites of two great ancient cities: the early Sumerian city Uruk and the early Elamite city Susa. The inscriptions, mainly accounts and receipts of various kinds, were written toward the end of the fourth millennium B.C. and soon afterward. After more than 100 years of scholarly effort all the systems of numbers and measures in these "proto-literate" texts have now been identified. *They turn out to include precursors of the later Sumero-Babylonian sexagesimal number system* (counting in 10's and 60's) and of our own decimal system (counting only in 10's). In addition they include a previously unrecognized system of capacity measures, used in all accounts dealing with barley, which in this early period was both the basic food grain and the currency.

The reader who would enjoy knowing about the proto-literate systems of numbers and measures will have to join me in a two-directional journey. We shall travel backward in time with respect to the historical record and forward from the past to the present with respect to the scholars who have studied the ancient tablets. *The reason we must do so is that the oldest tablets were buried the deepest and were therefore the last to be excavated and made available for study. The oldest tablets were also the most difficult to interpret.*

Let us take as our point of departure the Greek coastal island of Cos, some 20 miles northwest of Rhodes. There in about 340 B.C. the founder of a school of astrology, a Babylonian named Berossos, wrote a history of his homeland. In it he told his Greek readers that the numbers sossos (60), neros (600) and saros (3.600) occupied a special place in Babylonian arithmetic and astronomy. Practically nothing more was known about Babylonian numbers and measures for the next 2.200 years. Then in 1855 Sir Henry Rawlinson, one of the pioneers in the decipherment of cuneiform script, published a summary of the cuneiform numbers inscribed on a small clay tablet found at the site of the ancient Messopotamian city Larsa. Rawlinson realized, among other things, that the last two lines of the tablet stated in effect that “58 1 is the square of 59” and “1 is the square of 1”. He concluded that the tablet was the final part of an incomplete table of square roots, beginning with the square of 49 (equal to 2.401, or 40x60+1) and ending with the square of 60 (equal to 3.600, or 60x60). His interpretation, of course, was possible only if it was assumed that the numbers 60 and 60x60 were both represented by the same symbol, namely the symbol for the number 1.

Rawlinson drew the conclusion that the Babylonians had worked with a sexagesimal number notation of a quasipositional nature, in other words, a number notation in which the symbol for 1 also stood for the powers of 60 and for 10 times the powers of 60. He further concluded that the Babylonians did not have any special sign to represent zero.

Here it is necessary to briefly consider the relative merits of number systems with different bases. Let us begin with the so-called metric system, which is actually a family of interrelated systems of units for several kinds of measures. *The metric system owes its current acceptance to its structural simplicity and to the fact that it is constructed to match the base-10, or decimal, system used today for all kinds of routine computations.* Since its conception in France in the aftermath of the French Revolution it has gradually spread across the world.

The very length of time it has taken the metric system to gain general acceptance is proof of how difficult it is to suppress other "customary" systems of weights and measures. English examples of such systems include the sequences "mile", "furlong", "chain", "rod", "yard" , "foot", and "inch" for measures of length, "barrel", "bushel", "peck", "quart" and "pint" for measures of dry capacity and "ton", "hundred-weight", "pound" and "ounce" for measures of weight. *For that matter even the metric system has come to incorporate nondecimal systems:* the 12-month year, the 24-hour day, the 60-minute hour and the 60-second minute as units of time, and the 360-degree circle, with its subdivisions the 60-minute degree and the 60-second minute as units of angle. These customary measures can be traced back to classical Greek astronomy and beyond that to the general use of sexagesimal numbers for computation in Babylon and Sumer. *Many other customary systems of weights and measures, however, were doomed to be replaced by the metric system because they were insufficiently matched to the widely adopted decimal number system.*

Still, the survival of some customary systems has been partially the fault of the decimal number system itself. The decimal system has the weakness that its base of 10 is really too small. This will become more apparent when I give additional examples of computations within the framework of the sexagesimal system with its larger base: $60 = 3x4x5$ (as opposed to $10 = 2x5$). As will be seen, the base-60 system made it possible for the Sumerians' proto-literate predecessors to construct a family of nicely interrelated measure systems, with sequences of naturally occurring standard units that were easy to deal with in computations.

To find the correct interpretation of the system of cuneiform notations serving to represent sexagesimal (or base-60) numbers was relatively easy. It proved to be much harder to understand how the various systems of measures that appear clearly in many cuneiform inscriptions were constructed. Some decisive clues were offered by certain tablets known to scholars as school texts.

The copying of standard texts was an essential part of the school curriculum in Old Babylonian times (1900 to 1500 B.C.). Many of the texts were lists and tables: lists of geographic names, lists of the names of birds and fishes, lists of words in two languages, grammatical tables for the study of the difficult Summerian language and so on. Also copied were mathematical tables and lists explaining the structure of the Babylonian systems of measures and their representation in cuneiform script. By doing this kind of copying a student trained himself in cuneiform writing and at the same time accumulated a small personal library of tablets.

The first example of a table of measures to be described in a scholarly publication was a fragmentary tablet also uncovered at Larsa. The table was discussed by George Smith, a prominent student of cuneiform, in 1872, but its meaning was not fully understood until much later. On the left side of each column of the tablet is a systematically arranged sequence of linear measurements, expressed in standard units. The units are, from the smallest to the largest, the she (a grain), the shu-si (a finger), the kush (a cubit) and so on up to the beru, equal to 30x60x12 ($6x60^2$), or 21.600, cubits. On the right side of each column are the same linear measurements expressed as multiples of cubits in sexagesimal notation. For example, the line at the lower right labeled b in the reproduction of the tablet on page 81 reads "Two beru (equals) 12". It should be noted that beru is how the Babylonians pronounced the symbol for the Sumerian word danna (normally written as kas-gid, meaning "long way"). The 12, however, represents not 12 cubits but the much larger sum of 12 x 60^2 cubits. With a cubit being equal to about half a meter in length, the length of beru was more than 20 kilometers.

When another fragment of the same tablet was identified soon after the first fragment was found, it proved to contain an additional *metrological table of the same kind as the first, except that here the right side of each column was concerned with multiples of a nindan (equal to 12 cubits) in sexagesimal notation. Only much later did the study of Babylonian mathematical texts dealing with the computation of volumes make it clear that whereas the cubit was the basic unit for vertical measurements, the nindan was the unit for horizontal measurements.* Hence the smallest Sumero-Babylonian unit of area, the shar, was one square nindan. By the same token the smallest unit of volume, also called a shar, was the space enclosed by a bottom area of one square nindan that had sides one cubit high. This seemingly peculiar choice of units was actually quite practical because it usually excluded the need to count with small fractions of a volume unit.

These two metrological tables are eloquent witness to how well adapted the Sumero-Babylonian system of linear measurements was to the sexagesimal number system. Consider the conversion rules for units of the system of linear measures. Six she is equal to one shusi, 30 shu-si is equal to one kush, 12 kush is equal to one nindan, 60 nindan is equal to one USH and 30 USH is equal to one kas-gid (or beru). The information contained in this sequence of conversion rules can be condensed as follows: the "conversion factors" for the Babylonian linear system are 6, 30, 12, 60 and 30. Note that each one of these factors is also a numerical factor of the sexagesimal number system. By way of comparison, the Anglo-Saxon sequence from the inch to the mile involves the following conversion factors: 12, 3, 5 $\frac{1}{2}$, 4, 10 and 8. *Whatever the origins of the factors for this customary system, they are clearly in no way adopted to our decimal number system.*

Exercises

1. *In the text, translate the sentences italicized.*

2. *Explain the meanings of these words in which they occur in the text:*

record, scholar, power, matter, hard, respect, witness, conversion.

3. *Find words in the text with the same or similar meanings:*

1) difficult to do or understand
2) approximately

4. *Insert prepositions:*

to be equal ... about half a meter ... length, to be concerned ... smth, ... the same token, ... way of comparison, lists of words ... two languages, ... the same time, from the smallest ... the largest, ... the right side of each column, ... page 81, to deal ... smth, spread ... the world, to trace back ... smth, ... addition, with respect ... smth, ... other words, ... particular, ... effect.

5. *Give the English equivalents of the following:*

1) хорошо развитая система исчисления
2) десятичная система
3) в частности
4) счет на десятки
5) читателю было бы интересно узнать, что
6) вы вынуждены присоединиться ко мне
7) путешествие во времени
8) из прошлого в настоящее
9) самые сложные для понимания
10) в качестве отправной точки давайте возьмем
11) были представлены одним и тем же символом, а именно
12) особый символ для обозначения нуля
13) сейчас используется благодаря своей простоте
14) различные стандартные вычисления
15) доказательство того, как трудно конкурировать с другими системами
16) по этой причине
17) относительно легко
18) неотъемлемая составляющая школьной программы
19) изучение древних математических текстов прояснило
20) кроме того
21) этот кажущийся странным выбор
22) можно представить вкратце следующим образом
23) с которыми легко иметь дело при вычислениях

6. *Make up your own sentences on the basis of the following.*

1) They turn out to include ...
2) We shall travel backward in time ...
3) The reason we must do so is ...
4) In it he told his readers that ...
5) Practically nothing more was known about ...
6) He realized, among other things, that ...
7) He drew the conclusion that ...
8) Here it is necessary to briefly consider ...
9) Let us begin with ...
10) I give additional examples of ...
11) To find the correct interpretation ... was relatively easy.
12) It should be noted that ...

7. *Write a paragraph of about 150 words describing the merits of number systems with different bases.*

8. *Read the quotations below and answer the questions.*

Thomas Alva Edison (1847-1931), American inventor.

Genius is two percent inspiration, ninety-eight percent perspiration.

Henry Eyring (1901-81), American physical chemist.

A scientist's accomplishments are equal to the integral of his ability integrated over the hours of his effort.

1. Is it always so?

2. Could you name any outstanding person whose life seems to contradict Edison's words?

3. Did Edison and Eyring speak about the same?

VIII. A Mathematic Treasure in California

S.I.B.Gray, The Mathematical Intelligencer, Vol. 20, No. 2 (Spring 1998), pp. 41-46.

Does your hometown have any mathematical tourist attractions such as statues, plaques, graves, the cafe where the famous conjecture was made, the desk where the famous initials are scratched, birthplaces, houses, or memorials? Have you encountered a mathematical sight on your travels? If so, we invite you to submit to this column a picture, a description of its mathematical significance, and either a map or directions so that others may follow in your tracks.

Where should tourists go to see outstanding collections of historic mathematic books? Most of us would try Europe first, perhaps the Bibliotheque nationale de France in Paris, founded in 1367, before Gutenberg's time. At least since Mersenne, many outstanding mathematicians have enjoyed scholarly exchange in France. Thus, we might expect to find extensive collections of manuscripts and letters in Paris.

We could also travel across the Alps to Italy where we would find another candidate for the world's finest mathematics library in the Vatican. Founded in 1447, for handwritten manuscripts, the Biblioteca Apostolica Vaticana is especially strong in pre-Gutenberg materials. Although the Vatican has its antecedents in the Middle Ages and the Roman Empire, we note that the most famous center of learning in antiquity, the library in Alexandria, was completely destroyed by fire. *Thus, the Vatican Library, which might be expected to have had access to extremely rare ancient materials, found few surviving manuscripts.*

We might also expect to find great collections in Florence, the home of the Renaissance with its rebirth of scholarship founded on antiquity. Modern tourists can view the Biblioteca Medicea Laurenziana, founded in 1444 by Cosimo de'Medici. He chose Michelangelo as the architect to house the great family collection. But its collections were limited by the size of the room. When a room was filled, a new collection was started and moved elsewhere. Also, the Laurentian Library was organized while Florence was a seat of power and wealth, but long before the great mathematics of the 17th century.

What about the Teutonic world? The rise of Wissenschaften in the 19th century was coupled with the acquisition of major mathematics collections. But these collections were dispersed. With Gottingen, Munich, Berlin, and the more modern Max Planck Institutes, for example, all sharing a strong interest in a limited number of manuscripts, no single library is dominant today. Also, pillaging after warfare, dating at least to the 17th century, has diminished the chances of finding any one outstanding collection intact in Germany. Another factor is that 19th-century historians and librarians in Germany valued anitquity. A. Mommsen would write about the use of Roman coins, but probably never realized that his contemporaries - G. Cantor, Kronecker, and Hilbert - would have a major impact on future mathematical thought. *When D.E. Smith arrived in Germany in the 1880s to study the history of calculus, his mentor, M. Cantor, said, "Well, Mr. Smith, if I were you, I should not go back much father than Antipho and Bryso," - both of the fifth century B.C..* M. Cantor, as a historian, was not focused on the contributions of Barrow, Newton, and Leibnitz.

The English-speaking world is similar to the German world in its complexity. May excellent collections exist in Cambridge, at the Bodleian Library at Oxford, and at the British Museum, the Royal Society, and the Royal Institution in London. In Britain, one's choice of the premier library would depend on what century, what language, what translation, and whether one seeks books or manuscripts.

The Bodleian, founded in 1602, is the oldest library in England and thus has a special position. Posterity in general and mathematicians in particular owe a special debt of gratitude to the Bodleian and the Vatican libraries. They house copies of the two oldest editions of Euclid's Elements (888 and ca. 900 A.D., respectively). *Most mathematicians would agree that if one had to select a single title as the most important contribution from our discipline to civilization, the Elements would be named first.* Euclid is simply "the most successful textbook writer the world has ever known".

Are there any great collections in North America? In fact, there are quite a few. The Widener at Harvard, the Beinecke at Yale, the Regenstein at Chicago, and the Bancroft at Berkley all have wonderful collections. Columbia University has the awesome Plimpton Collection of clay tablets dating to the Old Babylonian period (ca. 1900 to 1600 B.C.). The Library of Congress has the Lessing J. Rosenwald Collection of 2,653 rare mathematics books. The Artemas Martin Collection at American University is rich in 19th-century textbooks. Many others deserve a visit. But a superb collection - arguably one of the finest in the world - is found in Southern California, of all places, near Los Angeles.

I. The Collector

The collection was built with wealth - the wealth of Henry E. Huntington, the nephew of a 19th-century railroad "robber baron." As the heir of one of Oscar Lewis's Big Four (Leland Stanford, Mark Hopkins, Charley Crocker, and Collis P. Huntington), Henry shared in the fortune derived from building and joining the Central Pacific and Union Pacific railroads in Promonotory, Utah Territory, 1869. Not one, but several spikes of gold, silver, and alloys were driven, representing the enormous wealth of joining the West Coast to the cities of the eastern United States. Symbolically, the wealth of the American West combined with the more refined tastes of the eastern elite.

When his uncle died, Henry was his favorite aide and confidant. He was managing the railroad-owned streetcar system that is so popular today with San Francisco tourists. Henry relocated to Los Angeles, and carried with him his knowledge of developing valley and foothill lands by controlling the rate and direction of public transportation. The business experience acquired in developing access to San Francisco's many hills worked with equal success in the vast Los Angeles basin.

Like his uncle, Henry continued to make money - and to hold on to it. He even outdid his uncle by consolidating the family fortune. Henry married his Uncle Collis's widow, a woman slightly his junior. Both of them admired Britain and the British passion for paintings, silver, porcelain, and furniture.

Henry was especially fond of rare and elegant book collections. From the time (1910) he moved into his newly built mansion in the Los Angeles suburb of San Marino until his death (1927), he bought en bloc every important library that came on the market in Britain. The power of his dollar so dominated the bibliographical markets of the world that American, British, and Continental estate owners simply emptied their shelves of hereditary collections. In this brief interval of time he acquired a Gutenberg Bible on vellum, the Ellesmere manuscript of Chaucer's Canterbury Tales, an unsurpassed number of original early Shakespearean editions, and an Audobon double-elephant folio of The Birds of America. He also inadvertently acquired mathematics titles. The British and Continental landed gentry had been purchasing mathematics publications since the 17th century. The libraries consolidated by Henry Huntington reflected the educated tastes of connoisseurs who were aware of the achievements of their contemporaries.

Later, Henry's collection was augmented by the efforts of astronomers who worked at Caltech's observatories. Both George

Ellery Hale and Edwin Hubble were devoted book collectors. Both were friends and colleagues of E.T. Bell. The Huntington's mathematics collection was Bell's primary source of scholarly references.

In addition, Bell used the Huntington as his benchmark in guiding other collectors. For example, Bell strongly advised the eminently successful Kentucky actuarial lawyer, William Marshall Bullitt, to try to acquire a copy of Niels Abel's Memoire, the now-famous eight page pamphlet published at Abel's own expense in 1824. Bullitt managed to purchase a copy from the widow of an actuarial professor in Oslo. The only other known surviving copies are in Gottingen and in the Mitta-Leffler Library in Stockholm.

II. Student Appreciation and Four Titles

What might a late-20th-century student of mathematics, knowing no Latin, be able to appreciate upon viewing the collection at the Huntington? The illustrations and many of the calculations show clearly that mathematics is a universal language that overcomes the limitations of words. The illustrations and diagrams, drawn by hand, are superb examples of craftsmanship. In most cases, the implications and detail are left to the reader, a procedure prevalent today as well.

Recently, my students, on viewing the Huntington collection in chronological order, remarked that Euler's Introductio was the first to look like a math book. By the 18th century, the equation and notation had evolved, as well as the printing techniques, to have math equations, not verbal explanations, embedded in the text. Calculus students will immediately recognize that Euler was working on infinite series.

Let's now examine four titles from the Huntington Collection to illustrate the value of showing our students works that in many instances predate Columbus's discovery of North America. The first is Ptolemy's Almagest, or "the greatest."

III. Almagest

Ptolemy wrote in Greek in the second century A.D. He produced the definitive Greek work on determining the location of the planets. His mathematics and its geocentric theory stood unchallenged for 1400 years until Copernicus proposed his heliocentric theory in 1543. *Viewing his page upon page of chord charts, representing essentially the first trigonometric tables, is an incredible experience.* The charts are a compilation of small, meticulously hand-written numbers. Each number records a value painstakingly determined.

This surviving copy uses the sexagesimal number system with Arabic [not Hindu-Arabic] numerals, the precursor of our modern counterparts of degrees, minutes, and seconds. *It is thought to have been produced in the south of France, but the monastery, or atelier, is unknown.* This translation is taken from Gerard of Cremona's work using an Arabic edition that entered Europe via the Moors in Spain.

The charts raise many questions. Mathematicians wonder what techniques were used to extract square roots. How were the calculations made? What theorems were applied? Did Ptolemy really have the modern sum and difference formulas for chords?

Claudius Ptolemy was a Roman citizen from a Greek family who is thought to have spent his entire adult life in Alexandria, Egypt; thus, Ptolemy most likely had a knowledge of Latin, Greek, and Aramaic. For careful study of his work, the sexagesimal, Greek, Roman, and modern number systems must be understood. Calculations for a circle are based not on 3600, but units which divide a circle into 120 parts. Values in modern trigonometric tables are ratios. Ptolemy, in the prevailing mode of his era, gave lengths of chords in a circle of radius 60, the base of his sexagesimal number system.

The Huntington's copy of the Almagest is illuminated with gold and color adornment. The vellum is not thick but has an unforgettable texture of strength tempered with the appeal of the finest translucent paper.

IV. Elements

The Huntington has more than 30 editions of the Elements, mostly in Latin, Greek, and English. But not all books have survived in all editions. Erhard Ratdolt's translation in Venice in 1482 was the first to be printed. Greek and Latin versions started appearing in many European countries. From the 17th to the 19th century, mathematicians in Britain often produced their own versions. Marginalia, illustrations, and comments often become incorporated with the text in subsequent editions. Todhunter's edition is an example which is still often found in public libraries.

For those familiar with a modern presentation of Euclid's axiomatic system, the 1482 edition contains some surprises. *The manuscript opens immediately with 23 definitions, with the first three being those of the point, line, and plane!* "A point is that which has no parts." "A line is without breadth." (What is a part? What is breadth?) Since David Hilbert's Grundlagen der Geometrie (1899), modern mathematicians have accepted these terms without definition in constructing an axiomatic system. Euclid never thought to do so.

Many - most - of the illustrations are immediately recognizable to any student of geometry, e.g., "diameter, circulus, major, minor, semicirculus, eqlaterus, perpendcularia [sic.]" Impressively, the Latin and illustrations of "punctus, linea, plana," are clear to English readers 500 years later.

On the second page, Euclid immediately sets forth ten principles of reasoning, his five mathematical postulates, and five "common notions". *In the lower left-hand margin is the illustration for the famous fifth, or parallel postulate. For a mathematician, the sight evokes the later drama of Saccheri, Gauss, Lobachevsky, Bolyai, and Riemann.* In later pages, two illustrations clearly communicate Euclid's "windmill" or "bride's chair" proof of the Pythagorean Theorem; also to be found is his "elefuga" or "pons asinorum" proof of the equality of the base angles of an isosceles triangle.

Some scholars admire a particular book of the Elements. E.T. Bell, for example, labeled Book V, or the rigorous introduction to the notion of continuity, a “masterpiece”. Others have expressed “fear” as well as admiration for Book X, which includes his geometric treatment of incommensurables, i.e. irrational numbers.

The Huntington has two copies of the first published English translation of Euclid’s Elements (London, 1570). The copies, which are slightly different, purport to be the work of Sir Henry Billingsley, who later became Lord Mayor. Billingsley’s Elements contains pop-up, three-dimensional models embedded in the text. Among others, a reader may assemble a pyramid, a tetrahedron, and perpendicular planes. Also, we find delightful English expressions, e.g., “A cube number is that which is equally equal equally or which is contained under three equal numbers”.

V. Analyse des Infiniment Petits

L’Hopital’s Analyse is the first differential calculus book to appear in print. In the text appears his eponymous rule, which is almost certainly the work of John Bernoulli. L’Hopital was an exceptionally clear and concise writer. There is no elaboration. A book that dominated mathematics as a principal text for most of one century needed only a few well chosen words - not equations - to make its points. This stands in sharp contrast to today’s calculus texts.

It is noteworthy that the only re-issued second edition among our selections is L’Hopital’s Analyse. The first edition was published in 1696, with the reprinted version appearing 12 years after his untimely death in 1704. All other books on our list are original first editions.

VI. Sumario Compendioso

The first mathematical work printed in the New World predates all North American settlement, i.e., Jamestown, Plymouth Colony, and Quebec City. The Sumario Compendioso was written in Spanish, not Latin, by Brother Juan Diez, “freyle”, and published by Juan Pablos Bressano, in Mexico City in 1556. We read, "El qual fue impresso en ja muy grande y muy leal ciudad de Mexico." The Sumario was the first textbook of any kind, other than religious instruction, published in the entire Western Hemisphere. Twenty-four of the 206 pages (103 folios) are devoted exclusively to arithmetic and algebra, while the rest cover the purchase price of various grades of silver, the purchase price of gold, percents, exchange rates, taxes, and other monetary affairs.

What type of algebra was being published in 1556? I quote from the first problem.

Primera quistion [cuestion] "Da me un numbero quadrado que restando del, 15, y 3/4 , quede fu propria rays [raiz]."

or, "Find a square number from which if $15\frac{3}{4}$ is subtracted, the difference is its own square root." The problem is followed by the "regla," or rule, and the proof, similar to the following:

Regla (Rule) Let the number be cosa. One half of cosa squared is $\frac{1}{4}$ of the zenso (quadrado). Adding 15 and $\frac{3}{4}$ to $\frac{1}{4}$ makes 16, of which the root is 4, and this plus $\frac{1}{2}$ is the root of the required number. Proof The "cosa" is $4\frac{1}{2}$ and the square of the cosa is the quadrado. $(9/2)2 = 20\frac{1}{4}$ $20\frac{1}{4} - 15\frac{3}{4} = 4\frac{1}{2}$ This will bear comparison with Brother Juan's far better known French contemporary, Francois Viete (1540-1603). Viete solved a quadratic $x^2 + ax = b$ by using $x = y - a/2$ to eliminate the linear term, and then also taking a square root. *Neither had the advantage of modern notation.*

Of the other two known copies of the Sumario, one is in the British Museum and the other at the Biblioteca Nacional in Madrid. Henry Huntington himself acquired the California copy in 1920, presumably not influenced by any philosophy of multiculturalism. David Eugene Smith of Columbia, a former MAA president, published a translation which you may borrow form UC Santa Barbara or UC Berkeley. [You may note Cajori borrowed the Berkeley copy several times.]

Also in Henry Huntinglon's day, and presumably heedless of gender equity issues, the Library acquired two titles by the Marquise du Chatelet. In fact, the Library has three copies of her famous translation of Newton's Principia. It also has Maria Agnesi's Instituzioni Analitiche, with her famous illustration of the versed sine curve, the "Witch of Agnesi".

VII. Visiting the Huntington

These editions are priceless and fragile. The Huntington Library seldom displays more than one or two mathematics and science books in one exhibit. Degradation associated with light and humidity is a particular concern. Security is another. At the present time should you, as a member of the general public, become one of the half million tourists who visit the Huntington Library each year, you would find only two works on display: Hubble's copy of Copernicus's De Revolutionibus Orbium Coelestium and Galileo's first illustrations of moon craters from Sidereus nuncius: Venice, 1610. To search more freely, you will need to make an appointment with the Library staff.

VIII. Acknowledgment

I thank Dr. Ronald Brashear, Curator, History of Science, Huntington Library, for his assistance and encouragement.

Exercises

1. *In the text, translate the sentences italicized.*

2. *What key words and phrases would you use to speak about Euclid's Elements?*

3. *Insert prepositions:*

to be strong ... smth, to focus ...smth, to be similar ... smth, to be fond ... smth, to be aware ... smth, to write ... Greek, to be familiar ... smth, to be clear ... smb, to be devoted ... smth.

4. *What do the abbreviations* **ca., A.D., B.C., e.g., i.e.** *stand for? Give their English equivalents.*

5. *Give the English equivalents of the following:*

1) бесценные издания
2) не выдерживать сравнение с
3) извлекать квадратный корень
4) взять книгу из библиотеки
5) страницы, посвященные исключительно алгебре
6) намного более известный современник
7) требуемое число
8) это сильно отличается от современных текстов
9) другие книги из списка
10) несоизмеримые иррациональные числа
11) первый напечатанный перевод
12) последующие издания
13) сохранившаяся рукопись
14) для тех, кто знаком с
15) на полях слева
16) работа по определению местонахождения
17) теория оставалась незыблемой
18) выдвинуть теорию
19) поставить много вопросов
20) выполнить вычисления
21) применить теорию
22) разделить круг на 120 частей
23) вычисления ясно показывают, что
24) зная о достижениях современников
25) редкие книги по математике

6. *Answer the following questions:*

a) What libraries are known to house extensive collections of historic mathematics books?

b) What was the most famous center of learning in antiquity?

c) What is Aramaic?

d) What was the connection of E.T. Bell with the Huntington?

e) Which books did E.T. Bell refer to as masterpieces?

f) What is the "Witch of Agnesi"?

g) It is known that the Huntington Library does not display a lot of mathematical books in one exhibit. Why?

7. *Describe four famous books from the Huntington collection examined in the article. Write a short summary.*

8. *Read the quotations below and answer the questions.*

Wystan Hugh Auden (1907-73), British-born American poet.

How happy the lot of the mathematician! He is judged solely by his peers, and the standard is so high that no colleague or rival can ever win a reputation he does not deserve.

Erick Christopher Zeeman (1925-), British mathematician.

The scientist has to take 95 per cent of his subject on trust. He has to because he can't possibly do all the experiments, therefore he has to take on trust the experiments all his colleagues and predecessors have done. Whereas a mathematician doesn't have to take anything on trust. Any theorem that's proved, he doesn't believe it, really, until he goes through the proof himself, and therefore he knows his whole subject from scratch. He's absolutely 100 per cent certain of it. And that gives him an extraordinary conviction of certainty, and an arrogance that scientists don't have.

1. Isn't mathematics a science? Consult an English-English dictionary.

2. What sciences does the author mean?

Nicholas Copernicus (1473-1543), Polish astronomer.

Mathematics is written for mathematicians.

Novalis (1772-1801), German Romantic poet.

Mathematics is the Life of the Gods.

3. Do you agree with Copernicus? Do you think there is any arrogance in his words?

4. Do you agree that Novalis expressed basically the same?

5. Do you consider mathematics to be superior to the sciences and humanities?

6. Do you think that mathematics is self-sufficient? If yes, is it always a good thing?

7. Which area of knowledge could contribute to mathematics?

IX. What Is the Difference between a Parabola and a Hyperbola?

Shreeram S. Abhyankar, The Mathematical Intelligencer, Volume 19, No. 4 (Fall 1988), pp. 36-43.

I. Parabola and Hyperbola

The *parabola* is given by the equation

$$Y^2 = X;$$

we can parametrize it by

$$X = t^2 \text{ and } Y = t.$$

The *hyperbola* is given by the equation

$$XY = 1;$$

we can parametrize it by

$$X = t \text{ and } Y = \frac{1}{t}.$$

Thus the parabola is a *polynomial curve* in the sense that we can paramatrize it by polynomial functions of the parameter t. On the other hand, for the hyperbola we need rational functions of t that are not polynomials; it can be shown that no polynomial parametrization is possible. Thus the hyperbola is not a polynomial curve, but it is a *rational curve. To find the reason behind this difference, let us note that the highest degree term in the equation of the parabola is Y^2, which has the only factor Y (repeated twice), whereas the highest degree term in the equation of the hyperbola is XY which has the two factors X and Y.*

II. Circle and Ellipse

We can also note that the *circle* is given by the equation

$$X^2 + Y^2 = 1;$$

we can parametrize it by

$$X = \cos\theta \text{ and } Y = \sin\theta.$$

By substituting $\tan\frac{\theta}{2} = t$ we get the *rational parametrization*

$$X = \frac{1-t^2}{1+t^2} \text{ and } Y = \frac{2t}{1+t^2},$$

which is not a *polynomial parametrization*. Similarly, the *ellipse* is given by the equation

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

and for it we can also obtain a rational parametrization that is not a polynomial parametrization. I did not start with the circle (or ellipse) because then the highest degree terms X^2+Y^2 (respectively, $(X^2/a^2) + (Y^2/b^2)$) do not have two factors, but we need complex numbers to find them.

III. Conics

In the above paragraph we have given the equations of parabola, hyperbola, circle, and ellipse in their *standard form.* Given the general equation of conic

$$aX^2 + 2hXY + bY^2 + 2fX + 2gY + c = 0,$$

by a linear change of coordinates, we can bring it to one of the above four standard forms, and then we can tell whether the conic is a parabola, hyperbola, ellipse, or circle. Now, the nature of the factors of the highest degree terms remains unchanged when we make such a change of coordinates. Therefore we can tell what kind of a conic we have, simply by factoring the highest degree terms. Namely, if the highest degree terms $aX^2 + 2hXY + bY^2$ have only one real factor, then the conic is a parabola; if they have two real factors, then it is a hyperbola; if they have two complex factors, then it is an ellipse; and, finally, if these two complex factors are the special factors $X \pm iY$, then it is a circle. *Here we are assuming that the conic in question does not degenerate into one or two lines.*

IV. Projective Plane

The geometric significance of the highest degree terms is that they dominate when X and Y are large. In other words, they give the behavior at infinity. To make this more vivid, we shall introduce certain fictious points, which are called the "points at infinity" on the given curve and which correspond to factors of the highest degree terms in the equation of the curve. These fictious points may be considered as "points" in the "projective plane." The concept of the projective plane may be described in the following two ways. A point in the *affine* (X, Y)-*plane*, i.e., in the ordinary (X, Y)-plane, is given by a pair (α, β) where α is the X-coordinate and β is the Y-coordinate. The idea of points at infinity can be made clear by introducing *homogeneous coordinates*. In this set-up, the old point (α, β) is represented by all triples $(k\alpha, k\beta, k)$ with $k \neq 0$, and we call any such triple $(k\alpha, k\beta, k)$ homogeneous (X, Y, Z)-coordinates of the point (α, β). This creates room for "points" whose homogeneous Z-coordinate is zero; we call these the *points at infinity*, and we call their totality the *line at infinity*. This amounts to enlarging the affine (X, Y)-plane to the *projective (X, Y,Z)-plane* by adjoining the line at infinity.

More directly, the projective (X, Y, Z)-plane is obtained by considering all triples (α, β, γ) and identifying proportional triples; in other words, (α, β, γ) and $(\alpha', \beta', \gamma')$ represent the same point if and only if $(\alpha', \beta', \gamma') = (\alpha, \beta, \gamma)$ for some $k \neq 0$; here we exclude the zero triple (0,0,0) from consideration. The line at infinity is now given by Z=0. To a point (α, β, γ) with $\gamma \neq 0$, i.e., to a point not on the line at infinity, there corresponds the point $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma})$ in the affine plane. In this correspondence, as γ tends to zero, $\frac{\alpha}{\gamma}$ or $\frac{\beta}{\gamma}$ tends to infinity; this explains why points whose homogeneous Z-coordinate is zero are called points at infinity.

To find the points at infinity on the given conic, we replace (X, Y) by $(X/Z, Y/Z)$ and multiply throughout by Z^2 to get the homogeneous equation

$$aX^2 + 2hXY + bY^2 + 2fXZ + 2gYZ + cZ^2 = 0$$

of the projective conic. On the one hand, the points of the original affine conic correspond to those points of the projective conic for which $Z \neq 0$. On the other hand, we put $Z = 0$ in the homogeneous equation and for the remaining expression we write

$$aX^2 + 2hXY + bY^2 = (pX - qY)(p^*X - q^*Y)$$

to get $(q, p, 0)$ and $(q^*, p^*, 0)$ as the points at infinity of the conic that correspond to the factors $(pX - qY)$ and $(p^*X - q^*Y)$ of the highest degree terms $aX^2 + 2hXY + bY^2$.

In the language of points at infinity, we may rephrase the above observation by saying that if the given conic has only one real point at infinity, then it is a parabola; if it has two real points at infinity, then it is a hyperbola; if it has two complex points at infinity, then it is an ellipse; and, finally, if these two complex points are the special points $(1, i, 0)$ and $(1, -i, 0)$, then it is a circle. At any rate, all the conics are rational curves, and among them the parabola is the only polynomial curve.

V. Polynomial Curves

The above information about parametrization suggests the following result.

THEOREM. A rational curve is a polynomial curve if and only if it has only one place at infinity.

Here *place* is a refinement of the idea of a point. *At a point there can be more than one place. To have only one place at infinity means to have only one point at infinity and to have only one place at that point. So what are the places at a point? To explain this, and having reviewed conics, let us briefly review cubics.*

VI. Cubics

The *nodal cubic* is given by the equation

$$Y^2 - X^2 - X^3 = 0.$$

It has a *double point* at the origin because the degree of the lowest degree terms in its equation is two. Moreover, this double point at the origin is a *node*, because at the origin the curve has the two *tangent lines*

$$Y = X \text{ and } Y = -X$$

(we recall that the tangent lines at the origin are given by the factors of the lowest degree terms). Likewise, the *cuspidal cubic* is given by the equation

$$Y^2 - X^3 = 0.$$

It has double point at the origin. Moreover, this double point at the origin is a *cusp*, because at the origin the curve has the only tangent line

$$Y = 0.$$

A first approximation to places is provided by the tangent lines. So the nodal cubic has two places at the origin, whereas the cuspidal cubic has only one. More precisely, the nodal cubic has two places at the origin because, although its equation cannot be factored as a polynomial, it does have two factors as a power series in X and Y; namely, by solving the equation we get

$$Y^2 - X^2 - X^3 = (Y - X(1+X)^{\frac{1}{2}})(Y + X(1+X)^{\frac{1}{2}}),$$

and by the binomial theorem we have

$$(1+X)^{\frac{1}{2}} = 1+(1/2)X+\ldots+\frac{(1/2)[(1/2)-1]\ldots[(1/2)-j+1)]}{j!}X^j+\ldots.$$

VII. Places at the Origin

Thus the number of places at the origin is defined to be equal to the number of distinct factors as power series, and in general this number is greater than or equal to the number of tangent lines. For example, the *tacnodal quintic* is given by the equation

$$Y^2 - X^4 - X^5 = 0,$$

which we find by multiplying the two opposite parabolas $Y \pm X^2 = 0$ and adding the extra term to make it irreducible as a polynomial. The double point at the origin is a *tacnode* because there is only one tangent line $Y = 0$ but two power series factors

$$(Y - X(1+X)^{\frac{1}{2}})(Y + X(1+X)^{\frac{1}{2}}).$$

So, more accurately, a cusp is a double point at which there is only one place; at a cusp it is also required that the tangent line meet the curve with *intersection multiplicity* three; i.e., when we substitute the equation of the tangent line into the equation of the curve, the resulting equation should have zero as a triple root. For example, by substituting the equation of the tangent line $Y = 0$ into the equation of the cuspidal cubic $Y^2 - X^3 = 0$, we get the equation $X^3 = 0$, which has zero as a triple root.

VIII. Places at Other Points

To find the number of places at any finite point, translate the coordinates to bring that point to the origin. To find the number of places at a point at infinity, *homogenize* and *dehomogenize*. For example, by homogenizing the nodal cubic, i.e., by multiplying the various terms by suitable powers of a new variable Z so that all the terms acquire the same degree, we get

$$Y^2Z - X^2Z - X^3 = 0.$$

By putting $Z = 0$ we get $X = 0$, i.e., *the line at infinity* $Z = 0$ meets the nodal cubic only in the point P for which $X = 0$. By a suitable dehomogenization, i.e., by putting $Y = 0$, we get

$$Z - X^2Z - X^3 = 0.$$

Now, P is at the origin in the (X,Z)-plane; the left-hand side of the above equation is *analytically irreducible*, i.e., it does not factor as a power series. Thus the nodal cubic has only one place at P. Consequently, in view of the above theorem, the nodal cubic may be expected to be a polynomial curve. To get an actual polynomial parametrization, substitute $Y = tX$ in the equation $Y^2+X^2-X^3 = 0$ to get

$$t^2X^2 - X^2 - X^3 = 0;$$

cancel the factor X^2 to obtain $X = t^2 - 1$ and then substitute this into $Y = tX$ to get $Y = t^3 - t$. Thus

$$X = t^2 - 1 \text{ and } Y = t^3 - t$$

is the desired polynomial parametrization. As a second example, recall that the nodal cubic $Y^2 - X^2 - X^3 = 0$ has two places at the origin, and the tangent line T given by $Y = X$ meets this cubic only at the origin. Therefore "by sending" T to infinity we could get a new cubic having only one point but two places at infinity; so it must be a rational curve that is not a polynomial curve. To find the equation of the new cubic, make the rotation $X' = X - Y$ and $Y' = X + Y$ to get $-X'Y' - (\frac{1}{8})(X' + Y')^3 = 0$ as the equation of the nodal cubic and $X' = 0$ as the equation of T. By homogenizing and multiplying by -8 we get $8X'Y'Z' + (X' + Y')^3 = 0$ as the homogeneous equation of the nodal cubic and $X' = 0$ as the equation of T. Labeling (Y', Z', X') as (X, Y, Z), we get $8ZXY + (Z + X)^3 = 0$ as the homogeneous equation of the new cubic and T becomes the line at infinity $Z = 0$. Finally, by putting $Z = 1$, we see that the new cubic is given by the equation

$$8XY + (1 + X)^3 = 0.$$

By plotting the curve we see that one place at the point at infinity $X = Z = 0$ corresponds to the parabola-like structure indicated by the hyperbola-like structure indicated by the two double arrows. Moreover, $Z = 0$ is the tangent to the parabola-like place, whereas $X = 0$ is the tangent to the parabola-like place. So this new cubic may be called the *para-hypal cubic*. To get a rational parametrization for it, we may simply take the vertical projection. In other words, by substituting $X = t$ in the above equation, we get $Y = -(1 + t)^3/8t$. Thus

$$X = t \text{ and } Y = \frac{-(1+t)^3}{8t}.$$

is the desired rational parametrization; it cannot be a polynomial parametrization.

IX. Desire for a Criterion

In view of the above theorem, it would be nice to have an algorithmic criterion for a given curve to have only one place at infinity or at a given point. Recently in [7] I have worked out such a criterion. See [2] to [6] for general information and [7] for details of proof; here I shall explain the matter descriptively. As a first step let us recall some basic facts about resultants.

X. Vanishing Subjects

In the above discussion I have often said "reviewing this" and "recalling that." Unfourtinately, reviewing and recalling may not apply to the younger generation. Until about thirty years ago, people learned in high school and college the two subjects called "theory of equations" and "analytic geometry." Then these two subjects gradually vanished from the syllabus. "Analytic geometry" first became a chapter, then a paragraph, and finally only a footnote in books on calculus. "Theory of equations" and "analytic geometry" were synthesized into a subject called "algebraic geometry." Better still, they were collectively called "algebraic geometry." *Then "algebraic geometry" became more and more abstract until it was difficult to comprehend.* Thus classical algebraic geometry was forgotten by the student of mathematics. Engineers are now ressurecting classical algebraic geometry, which has applications in computer-aided design, geometric modeling, and robotics. *Engineers have healthy attitudes; they want to solve equations concretely and algorithmically, an attitude not far from that of classical, or high-school algebra. So let us join hands with engineers.*

XI. Victim

Vis-a-vis the "theory of equations," one principal victim of the vanishing act was the resultant. At any rate, the Y-*resultant* $Res_Y(F,G)$ of two polynomials

$F = a_0Y^N + a_1Y^{N-1} + \ldots + a_N$ and $G = b_0Y^M + b_1Y^{M-1} + \ldots + b_M$

is the determinant of the N+M by N+M matrix

$$\begin{bmatrix} a_0 & a_1 & \dots & a_N & 0 & \dots & \dots & 0 \\ 0 & a_0 & \dots & \dots & a_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_0 & b_1 & \dots & b_M & 0 & \dots & \dots & 0 \\ 0 & b_0 & \dots & \dots & b_M & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

with M rows of the *as* followed by N rows of the *bs*. This concept was introduced by Sylvester in his 1840 paper. It can be shown that if $a_0 \neq 0 \neq b_0$ and

$$F = a_0 \prod_{j=1}^{N} (Y - \alpha_j) \text{ and } G = b_0 \prod_{k=1}^{M} (Y - \beta_k)$$

then

$$Res_Y(F,G) = a_0^M \prod_j (G(\alpha_j)) = (-1)^{NM} b_0^N \prod_k F(\beta_k) = a_0^M b_0^N \prod_{j,k} (\alpha_j - \beta_k).$$

In particular, F and G have a common root if and only if $Res_Y(F,G) = 0$.

XII. Approximate Roots

Henceforth let us consider an algebraic plane curve C defined by the equation

$$F(X,Y) = 0,$$

where F(X,Y) is a monic polynomial in Y with coefficients that are polynomials in X, i.e.,

$$F = F(X,Y) = Y^N + a_1(X)Y^{N-1} + \ldots + a_N(X),$$

where $a_1(X), \ldots, a_N(X)$ are polynomials in X. We want to describe a criterion for C to have only one place at infinity. *As a step toward* *this, given any positive integer D such that N is divisible by D, we would like to find the Dth root of F. We may not always be able to do this, because we wish to stay within polynomials. So we do the best we can.* Namely, we try to find

$$G = G(X,Y) = Y^{N/D} + b_1(X)Y^{(N/D)-1} + \ldots + b_{N/D}(X),$$

where $b_1(X), \ldots, b_(N/D)(X)$ are polynomials in X, such that G^D is as close to F as possible. More precisely, we try to minimize the Y-degree of $F - G^D$. It turns out that if we require

$$deg_Y(F - G^D) < N - (N/D),$$

then G exists in unique manner; we call this G the *approximate Dth root of F* and we donate it by $app(D, F)$. In a moment, by generalizing the usual decimal expansion, we shall give an algorithm for finding $app(D, F)$. So let us revert from high-school algebra to grade-school arithmetic and discuss decimal expansion.

XIII. Decimal Expansion

We use decimal expansion to represent integers without thinking. For example, in decimal expansion,

$$423 = (4 \times 100) + (2 \times 10) + 3.$$

We can also use binary expansion, or expansion to the base 12, and so on. *Quite generally, given any integer $P > 1$, every non-negative integer A has a unique P-adic expansion, i.e., A can uniquely be expressed as*

$$A = \sum A_j P^j \text{ with non} - \text{negative integers } A_j < P,$$

where the summation is over a finite set of non-negative integers j. We can also change bases continuously. *Namely, given any finite sequence $n = (n_1, n_2, \ldots, n_{h+1})$ of positive integers such that $n_1 = 1$ and n_{j+1} is divisible by n_j for $1 \leq j \leq h$, every non-negative integer*

A has a unique n-adic expansion; i.e., A can uniquely be expressed as

$$A = \sum_{j=1}^{h+1} e_j n_j,$$

where $e = (e_1, \ldots, e_{h+1})$ is a sequence of non-negative integers such that $e_j < (n_{j+1})/(n_j)$ for $1 \leq j \leq h$. In analogy with P-adic expansions of integers, given any

$$G = G(X,Y) = Y^M + b_1(X)Y^{M-1} + \ldots + b_M(X),$$

where $b_1(X), \ldots, b_M(X)$ are polynomials in X, every polynomial $H = H(X,Y)$ in X and Y has a unique *G-adic expansion*

$$H = \sum H_j G^j,$$

where the summation is over a finite set of non-negative integers j and where H_j is a polynomial in X and Y whose Y-degree is less than M. In particular, if N/M equals a positive integer D, then as G-adic expansion of F we have

$$F = G^D + B_1 G^{D-1} + \ldots + B_D,$$

where $B_1, \ldots, B_D$ are polynomials in X and Y whose Y-degree is less than N/D. Now clearly,

$$deg_Y(F - G^D) < N - (N/D) \text{ if and only if } B_1 = 0.$$

In general, in analogy with Shreedharacharya's method of solving quadratic equations by completing the square, for which reference may be made to [8] (and assuming that in our situation $1/D$ makes sense), we may "complete the Dth power" by putting $G' = G + (B_1/D)$ and by considering the G'-adic expansion

$$F = G'^D + B'_1 G'^{D-1} + \ldots + B'_D,$$

where $B'_1, \ldots, B'_D$ are polynomials in X and Y whose Y-degree is less than N/D. We can easily see that if $B_1 \neq 0$, then $deg_Y B'_1 < deg_Y B_1$. It follows that by starting with any G and repeating this procedure D times, we get the approximate Dth root of F. Again, in analogy with n-adic expansion, given any sequence $g = (g_1, \ldots, g_{h-1})$, where g_j is a monic polynomial of degree n_j in Y with coefficients that are polynomials in X, every polynomial H in X and Y has a unique *g-adic expansion*

$$H = \sum H_e \prod_{j=1}^{h+1} g_j^{e_j}, \text{ where } \mathrm{H_e} \text{ is a polynomial in X}$$

and where the summation is over all sequences of non-negative integers $e = (e_1, \ldots, e_{h+1})$ such that $e_j < n_{j+1}/n_j$ for $1 \leq j \leq h$.

XIV. Places at Infinity

As the next step toward the criterion, we associate several sequences with F as follows. *The case when Y divides F being trivial, we assume the contrary.* Now let

$$d_1 = r_0 = N, g_1 = Y, r_1 = deg_X Res_Y(G, g_1),$$

and

$$d_2 = GCD(r_0, r_1), g_2 = app(d_2, F), r_2 = deg_X Res_Y(F, g_2),$$

and

$$d_3 = GCD(r_0, r_1, r_2), g_3 = app(d_3, F), r_3 = deg_X Res_Y(F, g_3),$$

and so on, where we agreed to put

$$deg_X Res_Y(F, g_i) = -\infty \; if \; Res_Y(F, g_i) = 0$$

and

$$GCD(r_0, r_1, \ldots, r_i) = GCD(r_0, r_1, \ldots, r_j)$$

if $r_0, r_1, \ldots, r_j$ and integers and $j < i$ and $r_{j+1} = r_{j+2} = \ldots = r_i = -\infty$. Since $d_2 \geq d_3 \geq d_4 \geq \ldots$ are positive integers, there exists a unique positive integers h such that $d_2 > d_3 > \ldots >$

$d_{h+1} = d_{h+2}$. Thus we have defined the two sequences of integers $r = (r_0, r_1, \ldots, r_h)$ and $d = (d_1, d_2, \ldots, d_{h+1})$ and a third sequence $g = (g_1, g_2, \ldots, g_{h+1})$, where g_j is a monic polynomial of degree $n_j = d_1/d_j$ in Y with coefficients that are polynomials in X. Now, for the curve C defined by $F(X,Y) = 0$, we are ready to state the criterion.

CRITERION for having only one place at infinity. C has only one place at infinity if and only if $d_{h+1} = 1$ and $r_1 d_1 > r_2 d_2 > \ldots > r_h d_h$ and g_{j+1} is degree-wise straight relative to (r, g, g_j) for $1 \leq j \leq h$ (in the sense we shall define in a moment).

To spell out the definition of degree-wise straightness, for every polynomial H in X and Y we consider the g-adic expansion

$$H = \sum H_e \prod_{j=1}^{h+1} g_j^{e_j},$$

where H_e is a polynomial in X and where the summation is over all sequences of non-negative integers $e = (e_1, \ldots, e_{h+1})$ such that $e_j < n_{j+1}/n_j$ for $1 \leq j \leq h$. We define

$$fing(r, g, H) = max(\sum_{j=0}^{h} e_j r_j) \text{ with } e_0 = deg_X H_e,$$

where the max is taken over all e for which $H_e \neq 0 = e_{h+1}$; *here fing is supposed to be an abbreviation of the phrase "degree-wise formal intersection multiplicity," which in turn is meant to suggest some sort of analogy with intersection multiplicity of plane curves.* For $1 \leq j \leq h$, let $u(j) = n_{j+1}/n_j$ and consider the g_j-adic expansion

$$g_{j+1} = g_j^{u(j)} + \sum_{k=1}^{u(j)} g_{jk} g_j^{u(j)=k},$$

where $g_j k$ is a polynomial in X and Y whose Y-degree is less than n_j. We say that g_{j+1} is *degree-wise straight relative to* (r, g, g_j) if

$$(u(j)/k) fing(r, g, g_{jk}) \leq fing(r, g, g_{ju(J)}) = u(j)[fing(r, g, g_j)]$$

for $1 \leq k \leq u(j)$; the adjective *straight* is meant to suggest that we are considering some kind of generalization of Newton Polygon (for Newton Polygon, see [9]).

XV. Places at a Given Point

To discuss places of the curve C defined by $F(X,Y) = 0$ at a given finite point, we may suppose that the point has been brought to the origin by a translation and rotation of coordinates and that neither X nor Y divides F. By the Weierstrass Preparation Theorem, we can write

$$F(X,Y) = \delta(X,Y)F^*(X,Y),$$

where $\delta(X,Y)$ is a power series in X and Y with $\delta(0,0) \neq 0$ and F^* is a distinguished polynomial; i.e.,

$$F^* = F^*(X,Y) = Y^{N^*} + a_1^*(X)Y^{N^*-1} + \ldots + a_{N^*}^*(X)$$

and $a_1^*(X), \ldots, a_{N^*}^*(X)$ are power series in X that are zero at zero. By ord_X of a power series in X we mean the degree of the lowest degree term present in that power series. We also note that in the present situation, the approximate roots of F^* are monic polynomials in Y whose coefficients are power series in X. Now let

$$d_1 = r_0 = N^*, g_1 = Y, r_1 = ord_X Res_Y(F^*, g_1),$$

and

$$d_2 = GCD(r_0, r_1), g_2 = app(d_2, F^*), r_2 = ord_X Res_Y(F^*, g_2),$$

and

$$d_3 = GCD(r_0, r_1, r_2), g_3 = app(d_3, F^*), r_3 = ord_X Res_Y(F^*, g_3),$$

and so on, where we agree to put

$$ord_X Res_Y(F^*, g_i) = \infty \; if \; Res_Y(F^*, g_i) = 0$$

and

$$GCD(r_0, r_1, \ldots, r_i) = GCD(r_0, r_1, \ldots, r_j)$$

if $r_0, r_1, \ldots, r_j$ are integers and $j < i$ and $r_{j+1} = r_{j+2} = \ldots = r_i = \infty$. Since $d_2 \geq d_3 \geq d_4 \geq \ldots$ are positive integers, there exists a unique positive integer h such that $d_2 > d_3 > \ldots > d_{h+1} = d_{h+2}$. Thus we have defined the two sequences of integers $r = (r_0, r_1, \ldots, r_h)$ and $d = (d_1, d_2, \ldots, d_{h+1})$ and a third sequence $g = (g_1, g_2, \ldots, g_{h+1})$, where g_j is a monic polynomial of degree $n_j = d_1/d_j$ in Y with coefficients that are power series in X. For the curve C defined by $F(X,Y) = 0$, we are ready to state the main result of this section.

CRITERION for having only one place at the origin. C has only place at the origin if and only if $d_{h+1} = 1$ and $r_1 d_1 < r_2 d_2 < \ldots < r_h d_h$ and g_{j+1} is straight relative to (r, g, g_j) for $1 \leq j \leq h$ (in the sense which we shall define in a moment).

To spell out the definition of straightness, first note that in the present situation, the coefficients of a g-adic expansion are power series in X. Now for every polynomial H in Y with coefficients that are power series in X, we consider the g-adic expansion

$$H = \sum H_e \prod_{j=1}^{h+1} g_j^{e_j},$$

where H_e is a power series in X and where the summation is over all sequences of non-negative integers $e = (e_1, \ldots, e_{h+1})$ such that $e_j < n_{j+1}/n_j$ for $1 \leq j \leq h$. We define

$$fint(r, g, H) = min(\sum_{j=0}^{h} e_j r_j) \text{ with } e_0 = ord_X H_e,$$

where the min is taken over all e for which $H_e \neq 0 = e_{h+1}$; here fint is supposed to be an abbreviation of the phrase "formal intersection multiplicity," which in turn is meant to suggest some sort of analogy with intersection multiplicity of plane curves. For $1 \leq j \leq h$, let

$u(j) = n_{j+1}/n_j$ and consider the g_j-adic expansion

$$g_{j+1} = g_j^{u(j)} + \sum_{k=1}^{u(j)} g_{jk} g_j^{u(j)-k},$$

where we note that in the present situatition, the coefficients g_{jk} are polynomials of degree less than n_j in Y whose coefficients are power series in X. We say that g_{j+1} is *straight relative to* (r, g, r_j) if

$$(u(j)/k)fint(r, g, g_{jk}) \geq fint(r, g, g_{ju(j)}) = u(j)[fint(r, g, g_j)]$$

for $1 \leq k \leq u(j)$; again, the adjective *straight* is meant to suggest that we are considering some kind of generalization of Newton Polygon.

XVI. Problem

Generalize the above criterion by finding a finitistic algorithm to count the number of places at infinity or at a given point.

XVII. Example

To illustrate the criterion for having only one place at the origin, let us take

$$F = F(X, Y) = (Y^2 - X^3)^2 + X^p Y - X^7,$$

where p is a positive integer to be chosen. Now

$$F^* = F \text{ and } d_1 = r_0 = N^* = N = 4 \text{ and } g_1 = Y$$

and hence

$$Res_Y(F, g_1) = F(X, 0) = X^6 - X^7 \text{ and } r_1 = ord_X Res_Y(F, g_1) = 6.$$

Therefore,

$$d_2 = GCD(r_0, r_1) = GCD(4, 6) = 2$$

and hence

$$g_2 = app(d_2, F) = Y^2 - X^3 = (Y - X^{3/2})(Y + X^{3/2}).$$

Consequently,

$$\begin{aligned} Res_Y(F, g_2) &= F(X, X^{3/2})F(X, -X^{3/2}) = \\ &= (X^{p+3/2} - X^7)(-X^{p+3/2} - X^7) = -X^{2p+3} + X^{14}, \end{aligned} \quad \text{(IX.1)}$$

and hence

$$r_2 = ord_X Res_Y(F, g_2) = \left[\begin{array}{l} 14 \text{ if } p > 5; \\ 2p + 3 \text{ if } p \leq 5 \end{array}\right.$$

Therefore,

$$d_3 = \left[\begin{array}{l} 2 \text{ if } p > 5; \\ 1 \text{ if } p \leq 5 \end{array}\right. \text{ and } \left[\begin{array}{l} 1 \text{ if } p > 5; \\ 2 \text{ if } p \leq 5 \end{array}\right.$$

and

$$r_1 d_1 = \left[\begin{array}{l} 24 \leq 26 = (2p + 3)d = r_2 d_2 \text{ if } p > 5; \\ 24 \neq 22 \neq (2p + 3)d_2 = r_2 d_2 \text{ if } p \leq 5 \end{array}\right.$$

Now, if $p = 5$, then

$$g_{11} = 0, \text{ and } g_{21} = 0$$

and

$$g_{12} = X, \text{ and } fint(r, g, X^3) = 3r_0 = 12 = 2r_1$$

and

$$g_{22} = X^5Y - X^7, \text{ and } fint(r, g, X^5Y - X^7) = 5r_0 + r_1 = 26 = 2r_2,$$

and hence g_{j+1} is straight relative to (r, g, g_j) for $1 \leq j \leq 2$. Thus we see that if $p > 5$, then $h = 1$ and $d_{h+1} = 2$, whereas if $p < 5$, then $h = 2$ and $d_{h+1} = 1$ and $r_1 d_1 > r_2 d_2$; finally, if $p = 5$, then $h = 2$ and $d_{h+1} = 1$ and $r_1 d_1 \leq r_2 d_2$ and g_{j+1} is straight relative to (r, g, g_j) for $1 \leq j \leq 2$. Therefore, by the criterion we conclude that C has only one place at the origin if and only if $p = 5$.

Exercises

1. *In the text, translate the sentences italicized.*

2. *What key words and phrases would you use to describe the difference between a parabola and a hyperbola?*

3. *Identify the sentences with the Present Perfect. Explain why this form is used there.*

4. *Find two verbs in the text with the same or similar meaning to the following: to remove one thing and put something else in its place. Use them in your own sentences.*

5. *Insert prepositions:*

to be equal ... , to substitute smth ... smth, to multiply smth ... smth, places ... infinity, to correspond ... smth, ... other words, ... any rate, ... particular, ... analogy with smth, to be relative ... smth, the conic ... question, ... the one hand ... the other hand, to tend ... zero, a footnote in books ... calculus.

6. *a) Write down phrases from the text which are used to rephrase or give examples.*

b) What does abbreviation **i.e.** *stand for?*

7. *Give the English equivalents of:*

1) можно показать, что
2) давайте отметим, что
3) путем подстановки мы получаем
4) можно описать следующим образом
5) здесь мы подразумеваем, что
6) чтобы сделать это более наглядным, мы введем
7) эти точки можно рассматривать как
8) эту мысль можно пояснить, изображая

9) мы можем перефразировать наблюдение, сделанное выше, говоря, что

10) тогда и только тогда

11) напомним, что

12) точнее

13) хотя это уравнение не может быть представлено в виде

14) а именно: путем решения этого уравнения, мы получаем

15) когда мы подставляем уравнение касательной в уравнение кривой

16) полагая, что $Z = 0$, мы получаем

17) левая часть вышеупомянутого уравнения

18) чтобы получить желаемый результат

19) что, в свою очередь, означает

20) в степени меньшей, чем

8. *Compare different uses of* **to give** *and* **to do**. *Translate the sentences into Russian.*

a) The parabola is given by the equation $y^2 = X$.

b) In the above paragraph we have given the equations of parabola, hyperbola and ellipse in their standard form.

c) Given the general equation of conic ..., by a linear change of coordinates, we can bring it to one of the above four standard forms ...

d) We did not start with the circle ... because then the highest degree terms x^2+y^2 ... do not have two factors, but we need complex numbers to find them.

e) Here we are assuming that the conic in question does not degenerate into one or two lines.

f) More precisely, the nodal cubic has two places at the origin, because, although its equation cannot be factored as a polynomial, it does have two factors as a power series in X and Y; namely, by solving the equation we get ... .

9. *Write a summary of the text "Vanishing Subjects" in one or two sentences.*

10. *Read the quotations below and answer the questions.*

Enrico Fermi (1901-54), Italian physicist.

Young man, if I could remember the names of these particles, I would have been a botanist.

1. Are you surprised?

2. Is it really necessary to have factual knowledge?

3. Which is better for you – to derive formulas quickly when needed or know them by heart?

Charles Babbage (1792-1871), British mathematician.

A young man passes from our public schools to the universities, ignorant almost of the elements of every branch of useful knowledge.

4. Do you agree with the statement? Give reasons.

5. What kind of useful knowledge did Babbage speak about?

6. Has the situation changed since then?

7. What changes would you propose?

Niels Henrik David Bohr (1885-1962), Danish physicist.

How wonderful that we have met with a paradox. Now we have some hope of making progress.

George Bernard Shaw (1856-1950), British playwright, poet and critic.

Science is always wrong. It never solves a problem without creating ten more.

8. Try to confirm Bohr's idea by examples.

9. Do you agree that real progress can't be made without paradoxes?

X. Mathematics after Forty Years of the Space Age

Solomon W. Golomb, The Mathematical Intelligencer, vol. 21, number 4, Fall 1999, pp. 38-44.

When I was a graduate student at Harvard, in the early 1950s, the question of whether anything that was taught or studied in the Mathematics Department had any practical applications could not even be asked, let alone discussed. This was not unique to Harvard. *Good* mathematics had to be pure mathematics, and by definition it was not pemissible to talk about possible applications of pure mathematics.

This view was not invented by G.H. Hardy (1877-1947), but he was certainly one of its most eloquent and influential exponents. In *A Mathematician's Apology* (Cambridge U. Press, 1940) he wrote (p. 29), "Very little of mathematics is useful practically, and ... that little is comparatively dull"; and (p. 59), "The 'real' mathematics of the 'real' mathematicians, the mathematics of Fermat and Euler and Gauss and Riemann, is almost wholly 'useless' "; and (p. 79), "We have concluded that the trivial mathematics is, on the whole, useful, and that the real mathematics, on the whole, is not." In order to force external reality into his rhetorical model, Hardy decided to include leading theoretical physicists in his canon of "real" mathematicians, but to justify this by saying that their work had no real utility anyway. Thus, he wrote (p. 7l), "I count Maxwell and Einstein, Eddington and Dirac among 'real' mathematicians. The great modem achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present at any rate, almost as 'useless' as the theory of numbers."

Remember that this was in 1940; and Hardy also wrote (p. 80), "There is one comforting conclusion which is easy for a real mathematician. Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years." He also asserted (pp. 4l-42), "Only stellar astronomy and atomic physics deal with 'large' numbers, and they have very little more practical importance, as yet, than the most abstract pure mathematics." Today, 50 years after Hardy's death, it seems incredible that a book so at odds with reality was so influential for so many years.

It is ironic that Hardy's *Apology* was in fact not directed to mathematicians at all. After the dreadful carnage of World War I, and the realization that "the War to end Wars" hadn't really changed the world, pacifism was very widespread in England, and was effectively the established religion at Oxbridge between the Wars. The extreme attempts by Stanley Baldwin and Neville Chamberlain – who between them occupied 10 Downing Street from 1935 to 1940 – to avoid antagonizing Hitler can only be understood in this context. It was primarily to the non-scientists at Oxford and Cambridge that Hardy wanted to proclaim the *harmlessness* of mathematics. Hardy indicates that the *Apology*, in 1940, was an elaboration of his inaugural lecture at Oxford, in 1920, when the revulsion at the horrors of war would have been particularly vehement; and that he was reasserting his position that "mathematics [is] harmless, in the sense in which, for example, chemistry plainly is not".

Chemistry, responsible for the poison gases and disfiguring explosives of the Great War, is Hardy's chief example of a "useful" science, closely followed by Engineering, which does helpful things like building bridges, but destructive things as well, like designing warplanes and other munitions. In *A Mathematician's Apology*, Hardy is anxious to persuade his readership that "real" mathematics (especially the kind done by Hardy himself) is a noble aesthetic endeavor, akin to poetry, painting, and music, and has nothing in common with merely "useful" subjects like chemistry and engineering, which are also destructive in the service of warfare. A mere two years later, after the "blitz" bombing of London, Hardy's pacifist audience in England would have almost completely disappeared; but as a Mathematician's Manifesto, his *Apology* remained influential in mathematical circles for decades.

David Hilbert (1862-1943), regarded by many as the leading mathematician of the first four decades of the twentieth century, and who largely defined the agenda for twentieth-century mathematics with his famous list of twenty-three outstanding unsolved problems, presented at the International Congress of Mathematicians in Paris in 1900, largely shared and advocated the view advanced in Hardy's Apology. However, Hilbert's list had several problems motivated by numerical analysis, and one asking for a proper, rigorous mathematical formulation of the laws of physics. Coming just ahead of the discovery of relativity and quantum mechanics, this problem led to interesting mathematical work in directions Hilbert could not have anticipated, but in which he actively participated.

Another famous professor at Göttingen during the Hilbert epoch was Felix Klein, who had a much broader appreciation of applications. According to a famous story, a reporter once asked Klein if it was true that there was a conflict between "pure" and "applied" mathematics. Klein replied that it was wrong to think of it as a conflict, that it was really a complementarity. Each contributed to the other. The reporter then went to Hilbert, and told him, "Klein says there's no conflict between pure and applied mathematics." "Yes," said Hilbert, "of course he's right. How could there possibly be a conflict? The two have absolutely nothing in common." Since Hilbert, unlike Hardy, did work in areas of mathematics with obvious applications, and if the quote is authentic rather than apocryphal, the fundamental distinction he may have seen between pure and applied mathematics would likely have involved motivation - do we study it because it is beautiful or because it is useful?

Hilbert's illustrious contemporary and leading rival for the title of "greatest mathematician of the age" was Jules Henri Poincaré (1854-1912), a cousin of Raymond Poincaré (thrice Premier of France between 1912 and 1929, and President of France for seven years that included World War I). There is no question that Henri Poincare worked in some of the most obviously applicable areas of mathematics. Yet even Poincaré asserted: "The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living." Many leading scientists who have made major practical discoveries would share this view, but it is significantly different from Hardy's message. Nowhere does Poincaré suggest that applicable science, or useful mathematics, is in any way inferior, but rather that the systematic study of nature turns out to be inherently beautiful.

Through the ages, the very greatest mathematicians have always been interested in applications. That was certainly true of E.M. Bell's "three greatest mathematicians of all time": Archimedes, Newton, and Gauss. It was equally true of Euler, Lagrange, Laplace, and Fourier. Even in the first half of the twentieth century, it was true of Hermann Weyl, Norbert Wiener, and John von Neumann. As we come to the end of the twentieth century, the earlier insistence on the desired inapplicability of pure mathematics seems almost quaint, though one lingering legacy is that the label "applied mathematics" retains a pejorative taint and an aura of non-respectability in certain circles.

I want to examine the questions of when and how the concept of inviolable purity became entrenched in many departments of mathematics by the end of the nineteenth century, and what has happened in the past 40 years to weaken this presumption.

In the United States, the beginning of the modem research university dates back only to 1876, with the founding of Johns Hopkins, which was based on a German model that was only a few decades older. Prior to this period, the modern division of knowledge into departments and disciplines was much less rigid. In Newton's day, the term "natural philosophy" covered all of the natural sciences. The chair which Gauss (1777-1855) held at Göttingen was in Astronomy. Only when there were separate, clearly defined departments of mathematics was it necessary to invent a rationale to support their independence from either established or newly emerging fields which sought to apply mathematics. On the other hand, it was not necessary to justify the notion that every university needed a Department of Mathematics.

From the time of Plato's Academy, all through the Middle Ages, and into the rise of post-medieval universities, mathematics had always been central. The traditional "scholastic curriculum" consisted of two parts: the more elementary trivium, with its three language-related subjects – logic, grammar, and rhetoric; and the more advanced quadrivium, with its four mathematics – related subjects-geometry, astronomy, arithmetic (i.e., number theory), and music (i.e., harmonic relationships). At a time when Latin and Greek were indispensable parts of a university education, no one would have remotely considered eliminating mathematics as "impractical." Those students seeking a liberal university education, whether at Oxbridge in the U.K or in the Ivy League in the U.S., were not thought to be concerned with learning a trade and earning a living. That came much later. And high-budget research, with the concomitant requirement to set funding priorities, was not yet a part of the university scene.

So, in the late nineteenth century, university mathematics departments had a firm franchise to exist, and considerable latitude to define themselves. Much was happening in mathematics at that time (as well as ever since). The abstract approach was being applied, especially to algebra. The algebraic approach was being applied, especially to geometry and topology; analytic function theory was in full bloom; and a new standard of rigor had emerged. In many areas, mathematics was running so far ahead of applications that it was widely assumed that most of these fields would never have any. This was also true of certain classical areas, like number theory, which was developing rapidly as the beneficiary of new techniques from function theory and modern algebra, and had no foreseeable applications. Rather than be apologetic about the lack of applications for many areas, leading mathematicians and mathematics departments decided to turn this possible defect into a virtue. (In this, they anticipated a basic tenet of Madison Avenue: "If you can't fix it, feature it.") In fact, the best mathematics consistently found very important applications, but often not until many decades later. Riemann's "clearly inapplicable" non-Euclidean differential geometry (since everyone was certain that we live in a Euclidean universe), from the 1850s, became the mathematical basis for Einstein's General Theory of Relativity some 60 years later. Purely abstract concepts in group theory from around 1900 became central to the quantum mechanics of the 1930s and 1940s, and to the particle physics of the 1950s and onward. Finite fields, invented by Evariste Galois, who died in 1832, were considered the purest of pure mathematics, but since 1950 they have become the basis for the design of error-correcting codes, which are now used indispensably in everything from computer data storage systems to deep space communications to preserving the fidelity of music recorded on compact disks. George Boole's nineteenth-century invention of formal mathematical logic became the basis for electronic switching theory, from 1940 onward, and in turn for digital computer design.

Hardy's most precious area of inapplicable pure mathematics was prime number theory. Edmund Landau, in his Vorlesungen über Zahlentheorie ("Lectures on Number Theory", Leipzig, 1927), quotes one of his teachers, Gordan, as frequently remarking, "Die Zahlentheorie ist nutzlich, weil man nämlich mit ihr promovieren kann." ("Number Theory is useful because you can get a Ph.D. with it.") Today, the most widely used technique for "public key cryptography" is the so called RSA (Rivest, Shamir, and Adleman) algorithm, which depends on several theorems in prime number theory, and the observation that factoring a very large number into primes (especially if it is a product of only two big ones) is much harder than testing an individual large number for primality.

I think it is fair to say that in a very special sense, Number Theory has become a type of applied mathematics, and I'm not referring to number theory's applications in communication signal design and cryptography. Rather, I refer to the fact that Number Theory, which has rather limited methods of its own, has been the beneficiary of powerful applications to it from analytic function theory, from modern algebra, and most recently from algebraic geometry, as with Andrew Wiles's proof of "Fermat's Last Theorem."

In 1940, topology would have been high on most people's lists of inapplicable mathematics. Within topology, knot theory would have seemed particularly useless. Yet today, knot theory has extremely important applications in physics (to both quantum mechanics and superstring theory) and in molecular biology (to the knotted structures of both nucleic acids and proteins). The topology of surfaces is also much involved in superstring theory, including the structures which superstrings may take in multidimensional spaces. Even graph theory, the "trivial" one-dimensional case of topology, has blossomed into a major discipline where the boundary between "pure" and "applied" is virtually invisible. Until recently, tiling problems were largely relegated to the domain of "recreational mathematics" (an obvious oxymoron to most non-mathematicians, but a pleonasm to true believers). Then, a decade or so ago, Roger Penrose's work on small sets of tiling shapes which can be used to tile the entire plane, but only non-periodically, was found to underlie the entire vast field of "quasicrystals."

I could give many, many more examples of how topics and results from the "purest" areas of mathematics have found very important applications, but I believe I have made my point. It may still be necessary for some Mathematics Departments to defend themselves from being turned into short-term providers of assistance to other disciplines which are consumers rather than producers of mathematics; but the basic principle that good "pure" mathematics is almost certain to have very important applications eventually is now widely recognized. For most mathematicians today, the distinction that matters is between good mathematics and bad mathematics, not between pure mathematics and applied mathematics. To be fair to Hardy, this was the distinction he was trying to make in *A Mathematician's Apology* between "real" mathematics and "trivial" mathematics. Where he went off the deep end was in trying to insist that real mathematics is useless, and that useful mathematics is trivial.

Like many of my generation, I was attracted to mathematics not by Hardy's Apology, but by E.M. Bell's *Men of Mathematics*, and my early interest in number theory was partly motivated by the accessibility of the subject. The first mathematics book I ever bought, with my own almost non-existent disposable income while I was still in high school, was Carmichael's thin volume *Theory of Numbers*, in hard cover. Two years later, I was systematically reading Landau's *Vorlesungen über Zahlentheorie*, still on my own. When I came to Harvard as a graduate student, I already assumed that I would do a thesis in prime number theory. This was a respectable branch of mathematics at Harvard, although none of the faculty there specialized in it. David Widder, who had recruited me to be his student my very first day of classes at Harvard, and who included an analytic proof of the Prime Number Theorem in his book *The Laplace Transform*, was happy to sponsor my efforts. He had spent time as a post-doc of Hardy and Littlewood in Cambridge, where the highlight of his sojourn was attending a cricket match seated between these two famous gentlemen.

For help and inspiration, I drove to the Institute for Advanced Study in Princeton one morning in October, 1953, quite unannounced, and went to see Atle Selberg. I had discovered an identity involving von Mangoldt's lambda function, which I thought could be useful in analytic number theory. My identity looked slightly like Selberg's Lemma in his famous Elementary Proof of the Prime Number Theorem. His first reaction was to think that my identity was false. Trying to disprove it, he convinced himself in the next ten minutes that it was true. He then spent the rest of the day with me, exploring ways I proposed to use this identity, and making many helpful suggestions. I learned only later that Selberg had a reputation for being totally reserved and unapproachable.

The other number-theorist who was very helpful was Paul Erdős, who was always totally approachable if you wanted to talk about mathematics. I learned only years later of the supposed feud between Selberg and Erdős. Years later still, in November, 1963, I saw them both in the same room at a Number Theory Conference at Caltech. It was the day John Kennedy was assassinated. I remember the session chairman announcing the news flash that JFK was dead. After no more than half a minute, the meeting resumed as before. I suspect it was the only activity in the whole country that Friday afternoon that wasn't shut down. I mentioned this long afterward to an insightful mathematician friend, and I commented that these mathematicians hadn't reacted to the news of JFK's death. "No, you don't understand," he told me. "They're mathematicians. That was their reaction."

I could have finished up at Harvard in the spring of 1955, in time for my twenty-third birthday, but having been awarded a Fulbright fellowship for study in Norway, I decided to finish my thesis writeup there. I wasn't even sure who was still active in Norway when I applied for the fellowship. There were many famous Norwegian mathematicians, but I knew that Niels Henrik Abel, Sophus Lie, and Axel Thue were long dead, and that several others, including Osvald Veblen, Einar Hille, Øystein Ore, and of course Atle Selberg had resettled in the United States. From Landau's book I had learned about Viggo Brun's sieve method in prime number theory, but for all I knew Brun was also long dead. Fortunately this was not the case. Brun turned 70 the month I arrived, in June, 1955, but he did not retire until a year later, and he lived well beyond age 90. From Landau's austere *Satz, Beweis* approach, I could prove Brun's Theorem, that the series consisting of the reciprocals of the twin primes is either finite or convergent, but I had no understanding of what motivated it, or why it worked. It was only when Brun explained his method to me that it made sense. I included a sieve-derived result in my thesis, which I also published in Mathematica Scandinavica. A few years later, Erdős got an improvement on my result, which he published in the Australian Journal of Mathematics, in a paper titled "On a Problem of S. Golomb." Dozens of people have published papers titled "On a Problem of Erdős," but since Erdős published this one, "On a Problem of S. Golomb," I claim that my "Erdős number" is minus one.

For four summers while a graduate student, I worked at the Martin Company, now part of Lockheed-Martin, and became quite interested in mathematical communication theory, including Shannon's Information Theory, and especially "shift register sequences," which were of interest for a variety of communications applications, but which I discovered were modeled by polynomials over finite fields. Shannon's epic paper, "A Mathematical Theory of Communication," was published in 1948, the year after Hardy died, but had Hardy read and understood it, he would have called it "real" mathematics, secure in the belief, that it was not really "useful." After all, Shannon gave existence proofs that codes could be constructed arbitrarily close to certain bounds, with no hint of how to find such codes. But what has happened in the past fifty years is that mathematicians have worked closely with communications engineers to develop Information Theory and Coding in a way that is simultaneously first-rate "real" mathematics and eminently practical and useful engineering. In several prominent cases, the same individual has spanned the entire range from developing the theoretical mathematics to designing the practical hardware.

I eventually discovered that an important early paper on linear recurrences over finite fields was published in 1934 by Øystein Ore, but if not for the wide range of applications to several areas of technology, including communications, I don't think we would have a subject classification today in *Mathematical Reviews* called "Shift Register Sequences," corresponding to literally hundreds of published research papers. Perhaps Hardy would have been disturbed to learn how practical the properties of finite fields have become-but then, none of his sacred cows has remained untainted.

When I returned from Norway in the summer of 1956, I came to Southern California to work in the Communication Research Group at the Jet Propulsion Laboratory (JPL), in Pasadena. This job, which grew out of the interest I had developed in mathematical communications during my summer jobs at the Martin Company, enabled me to continue my search for applications of "useless" mathematics to practical communications problems. Over the next seven years I formed and headed a group of outstanding young researchers who developed the systems that made it possible to communicate with space vehicles as far away as Neptune (three billion miles from Earth) and beyond.

In 1956, NASA did not yet exist, and JPL was funded by the Ordnance Command of the U.S. Army. They also supported Wernher von Braun's group at Redstone Arsenal in Huntsville, Alabama. The U.S. Army was prepared to launch a small artificial satellite in September, 1956, thirteen months before Sputnik, using a Redstone missile as the launch vehicle, and a small JPL-built payload with a radio transmitter, but General John Bruce Medaris, head of the Army Ballistic Missile Agency, was unable to get the permission of the Eisenhower administration to proceed. Unaware that we were in any kind of race with the Soviet Union, the Eisenhower administration had decided that the U.S. space program should be peaceful, and therefore should not use an Army missile as the launch vehicle. Instead, we had something called Project Vanguard under development, for which the launch vehicle would be a Navy missile!

The Soviet Union's launch of Sputnik l, on October 4, 1957, took the world by surprise. It was visible to the naked eye in the night sky, and a fairly simple radio receiver could pick up its "beep-beep" signal. Around November 12, 1957, General Medaris not only had pemission to launch a satellite using Army vehicles, but he had orders to proceed as quickly as possible. Meanwhile, on December 6, 1957, the first launch of a Vanguard satellite was attempted. With the entire world press corps watching, there was a spectacular explosion on the launch pad. The vehicle was consumed in flames from the bottom upward.

Eighty days after the authorization to proceed, the Army satellite was ready for launch. The first stage was a liquid-fuelled elongated Redstone rocket, from Huntsville. The second stage was a cluster of eleven solid-fuel Sergeant missiles from JPL. The third stage used three Sergeant missiles from JPL. And the fourth stage, built at JPL, was a cylinder about 5 feet long and 8 inches in diameter, packed with electronic equipment, and sitting atop a final Sergeant missile. This configuration had never been tested, but the launch of Explorer I, on January 31, 1958, was a success on the very first try. I had a lab at JPL at that time, where I studied the properties of shift register sequences experimentally. During the weeks leading up to Explorer I, a graduate student of James Van Allen was assembling a radiation detector in my lab, with the assistance of my technicians. It was this detector, flying on Explorer I, that "discovered" what came to be known as the Van Allen Radiation Belts around the earth. My own special assignment was to participate in the "early orbit determination" of the satellite.

After Explorer I was launched from Cape Canaveral, no signal was picked up by the down-range station on the Caribbean island of Antigua. We did not know that the signal had been successfully detected and recorded at our stations in Nigeria and Singapore until a few days later, when we were notified by air mail! Amateur radio groups in Australia and Hawaii reported nothing. We were understandably worried that we had lost our satellite. We had three tracking stations widely spaced in Southern California, connected to JPL only by ordinary telephone lines. The nominal time for the satellite to come into radio contact over California came and went, with no detection by any of our stations. Three minutes passed, then another three minutes, and still no detection. There were many long faces in our orbit determination room at JPL. Then, about eight minutes late, all three of our tracking stations called in almost simultaneously. Explorer I was alive and well. One of the upper stages of the launch rocketry of Explorer I had over-performed, slightly enlarging the orbit and lengthening the period, and incidentally increasing its lifetime in orbit. That was an exciting time to be at JPL.

While I was at JPL, I learned the real distinction between "pure research" and "applied research." In 1959, the Laboratory director, Dr. William H. Pickering, decided to form an ad hoc committee, with representatives from all over JPL, to report on the "research environment" and what could be done to improve it; and he appointed me to chair it. I discovered that every member had strong opinions about what was pure research and what was applied research, and surprisingly, it had absolutely nothing to do with the subject matter. It all boiled down to this. What you want to work on is pure research. What your boss wants you to work on is applied research.

Sputnik shocked the American public. The notion that the Soviets were ahead of us in rocketry contradicted a basic tenet in Vannevar Bush's influential book *Modern Arms and Free Men*, which argued that a closed non-democratic society like the Soviet Union couldn't possibly develop armaments and advanced weapons as well as we could in the U.S. Bush had developed a very early analog computer, called the "Bush differential analyzer," prior to World War II. During the war, he was Roosevelt's chief advisor on scientific and technological matters. He delivered the letter to FDR, drafted by Szilard and Wigner, and signed by Einstein, which led to the Manhattan Project and the Atomic Bomb. Beyond that, most of the things Vannevar Bush recommended - at least, the ones I am aware of were ill-advised. For example, because of his bias in favor of analog computing, he delayed research on digital computers until after the war. In Modern Arms and Free Men, he not only asserted that the Soviets couldn't possibly come up with first-rate weapons, but also that neither we nor they could ever develop intercontinental ballistic missiles (lCBMs). In 1957, I got a letter from Bush on his letterhead stationery as Chairman of the Board of MIT. Martin Gardner had run an article in Scientific Amencan about my "polyominoes," and included my proof that a particular covering of the checker-board with pieces of a certain shape was impossible, which was shown by coloring the board in a particular way. Bush wanted to know why the result would still hold if you didn't color the board in that particular way! Naturally I wrote a very polite and patient reply. (By the way, "Polyominoes" is also a subject classification in MR.) But perhaps I am too harsh on Dr. Bush. A newly published Bush biography credits him with creating the post-World-War-II structure of government funding for university research, which puts many of us in his debt.

The extent to which we were behind the Soviets in the development of large missiles was mostly a political matter, and secondarily an engineering issue. Basic science was not really involved at all. Nonetheless, both the public and the politicians were convinced that a much greater commitment to the support and funding of research, especially university research, in all the basic sciences, including mathematics, was an urgent national priority. I believe it is very fortunate that this happened, and that it contributed significantly to the U.S. winning the Cold War some 30 years later, but it had no relationship to the issue of whether the U.S. was behind in rocketry.

As you will remember, in *A Mathematician's Apology*, Hardy had contended that "real" mathematics is much more similar to poetry and painting that it is to chemistry or engineering. That is something that many of us, as mathematicians, might still like to believe; but the new government funding didn't extend to poetry and painting. The rationale for including mathematics in the new governmental largess required a commitment to the principle that basic research in mathematics, like basic research in chemistry or engineering, will ultimately have practical, beneficial consequences. Suddenly, there was a reason for trying to show that one's mathematics had practical uses and implications. This was not the only reason for the change in attitude about whether good mathematics could be useful, but it certainly played a part.

Another major influence has been the development of digital technology, which has placed new emphasis on areas of discrete mathematics that were previously considered inapplicable-like finite fields, which I've already mentioned. Then there is Computer Science itself, which asks questions in pure mathematics like finding the computational complexity of various procedures, which turns out to be extremely practical. Another development has been Shannon's mathematical theory of communication, which asks questions motivated by applications, but which are more abstract mathematically than anything in physics. An atom, an electron, a photon, or a quark – these are all entities in the physical world whose behavior the physicist attempts to model. But Shannon's "bit of information" is a purely mathematical concept. It has no mass, no spin, no charge, no momentum – and yet the issues involved in measuring information in bits, in storing information, in moving information from one place to another are so important that we are told that we live in the "Age of Information."

It is also true that science and engineering have changed dramatically in the fifty years since Hardy's death. Semiconductors and lasers make nontrivial use of quantum mechanics, and the people who study them are not restricted to using what Hardy derisively referred to as "school mathematics." Biology at the University level has progressed from butterfly collecting to genome sequencing. It is not at all clear-cut whether "control theory" is a topic in mathematics or a branch of engineering.

I've never called myself an "applied mathematician." When I'm doing mathematics as mathematics I am a mathematician. When I'm focusing on applications to communications, I'm a communications engineer. For the first several years that I worked on mathematical communications problems, I didn't even realize that there were good journals in which new results in these areas could and should be published. That was a lingering aftereffect of my Hardy-style brainwashing. I hope I've finally outgrown it. In fact, I hope we've all outgrown it. Mathematics isn't "good" just because it's inapplicable, and it isn't "bad" just because it is.

In fairness to Hardy, there are many things in *A Mathematician's Apology* with which most mathematicians will agree or identify. Hardy asserts that mathematicians are attracted to the subject by its inner beauty, rather than by any overwhelming desire to benefit humanity. Most mathematicians I know would agree with that. Even more important, Hardy identifies himself (p. 63) as a Realist (as the term is used in Philosophy) about mathematics. "I will state my own position dogmatically .... I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations. This view has been held, in one form or another, by many philosophers of high reputation from Plato onwards .... " The great majority of mathematicians share this view about mathematics. Plato went overboard, trying to extend mathematical reality to physical reality. Immanuel Kant explicitly distinguished between the "transcendental reality" of mathematics and the (ordinary) reality of the physical universe. My own version of this distinction is that if the Big Bang had gone slightly differently, or if we were able to spy on an entirely different universe, the laws of physics could be different from the ones we know, but 17 would still be a prime number. I recently found a very similar view attributed to the late great Julia Robinson (1919-1985) in the biography *Julia, a Life in Mathematics*, by her sister, Constance Reid. “I think that I have always had a basic liking for the natural numbers. To me they are the one real thing. We can conceive of a chemistry that is different from ours, or a biology, but we cannot conceive of a different mathematics of numbers. What is proved about numbers will be a fact in any universe.” This is also reminiscent of the famous dictum of Leopold Kronecker (1823-1891): “Die ganzen Zahlen hat Gott gemacht; alles anderes ist Menschenwerk.” (“God made the whole numbers; everything else is the work of man.”)

Platonism (i.e. “Realism”) about mathematics has dissenters. Some who, in my view, are overly influenced by quantum mechanics, would argue that 2^{-1}, where P is some very large prime number, is neither prime nor composite, but in some intermediate “quantum state,” until it is actually tested. Of course, the Realist view is that it is already one or the other (either prime or composite), and we find out which when we test it. Even less palatable to most mathematicians is the “post-modern” criticism of all of “science,” that it is just another cultural activity of humans, and that its results are no more absolute or inevitable than works of poetry, music, or literature. The extreme form of this viewpoint would assert that “$4 + 7 = 11$” is merely a cultural prejudice. I will readily concede the obvious: it requires a reasoning device like the human brain (or a digital computer) to perform the sequences of steps that we call “mathematics”. Also, culture can play an important role in determining which mathematical questions are asked, and which mathematical topics are studied. (Our widespread use of the decimal system is undoubtedly related to humans having ten fingers.) What I will not concede is that, if the same mathematical questions are asked, the answers would come out inconsistently in another culture, on another planet, in another galaxy, or even in a different universe. For example, the Greeks were interested in “perfect numbers,” numbers like 6 ($= 1 + 2 + 3$) and 28 ($= 1 + 2 + 4 + 7 + 14$) which equal the sum of their exact divisors (less than the number itself). I can readily imagine a “civilization” with advanced mathematics in which the notion of “perfect numbers” was never formulated. What I cannot imagine is a civilization in which perfect numbers were defined the same way as we do, but where 28 was no longer perfect.

It was Isaac Newton who wrote, “I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great Ocean of Truth lay all undiscovered before me.” It is hard to give a better formulation of the Realist view of mathematical truth adhered to by most mathematicians.

Newton is firmly entrenched in the pantheons of both mathematics and physics. He certainly erected no artificial boundaries between theory and applications. In the three centuries since he published the *Philosophiae naturalis principia mathematica* (the *Principia*, for short), we have, with electronmagnetism and relativity and quantum mechanics, waded deeper into that Ocean of Truth; but Newton’s laws of motion and gravitation were actually sufficient for the launching of Sputnik and Explorer. Today, 40 years after those satellites first circled the earth, most mathematicians - myself included - have moved farther from Hardy’s outlook and closer to Newton’s.

Exercises

1. *Translate the following sentences into Russian:*

1) Nowhere does Poincaré suggest that applicable science, or useful mathematics, is in any way inferior, but rather that the systematic study of nature turns out to be inherently beautiful.

2) It was Isaac Newton who wrote, “I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing in the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great Ocean of Truth lay all undiscovered before me.”

3) Only when there were separate, clearly defined departments of mathematics was it necessary to invent a rationale to support

their independence from either established or newly emerging fields which sought to apply mathematics.

2. *Specify the italicized verbal forms in the sentences below and translate them into Russian:*

1) This job, which *grew* out of the interest I *had developed* in mathematical communications during my summer jobs at the Martin Company, *enabled* me to continue my search for applications of "useless" mathematics to practical communications problems.

2) ... Hilbert's list had several problems *motivated* by numerical analysis, and one *asking* for a proper, rigorous mathematical formulation of the laws of physics.

3) *Coming* just ahead of the discovery of relativity and quantum mechanics, this problem led to interesting mathematical work in directions Hilbert *could not have anticipated*, but in which he actively participated.

4) Since Hilbert, unlike Hardy, *did work* in areas of mathematics with obvious applications, and if the quote is authentic rather than apocryphal, the fundamental distinction he *may have seen* between pure and applied mathematics *would* likely *have involved* motivation - do we study it because it is beautiful or because it is useful?

5) The abstract approach *was being applied*, especially to algebra. The algebraic approach *was being applied*, especially to geometry and topology; analytic function theory was in full bloom; and a new standard of rigor *had emerged.*

6) *I could have finished up* at Harvard in the spring of 1955, in time for my twenty-third birthday, but *having been awarded* a Fulbright fellowship for study in Norway, I decided to finish my thesis writeup there.

7) Biology at the University level *has progressed* from butterfly collecting to genome sequencing.

8) Newton's laws of motion and gravitation were actually sufficient for the *launching* of Sputnik and Explorer.

9) Through the ages, the very greatest mathematicians *have* always *been interested* in mathematics.

3. *Give the English equivalents of the following Russian terms. Make sure that you know how to spell them.*

Прикладная математика, общая теория относительности, квантовая механика, физика, теория чисел, простое число, теория функций, дифференциальная геометрия, физика частиц, теория поля, конечная группа, конечное поле, теория групп, код с исправлением ошибок, раскладывать число на простые множители, простота (примарность), теория узлов, алгебраическая геометрия, теория суперструн, многомерное пространство, теория замощения, многочлен, линейная рекурсия, граница, доказательство существования, цилиндр, дюйм, фут, показатель степени.

4. *Add the missing letters to complete the words below. Use your dictionary to check your answers, if necessary.*

Rese. . . rch, s. . . multan. . . ously, inc. . . dentally, mat. . . ematics, pol. . . nomial, f...nite, f...eld, proo..., exist...nce, lin...ar, rec...rrence, ser...es, rec...procal, converg...nt, virt...ally, t...ling, mult...dimen... ional, cr...ptography, separ...te, dis...ipl...ne, ri...id, qu...int.

5. *Using your English-English dictionary, say what words the following derive from and specify their suffixes and prefixes. Mind the pronunciation.*

Influential, harmlessness, foreseeable, primality, accessibility, understandably, secondarily, largess, aftereffect.

Using the word in capitals form its derivatives to complete the sentences below:

APPLY

1. There is no question that Henri Poincaré worked in some of the most obviously ... areas of mathematics.

2. The ... of the new method was clearly understood by the leading researchers.

3. In 1940, topology would have been high on most people's lists of ... mathematics.

NB! There are the so-called blends, or *portmanteaux words*, which take two lexemes and combine them to make one. Usually the components of a blend are recognizable. Here are some examples.

breakfast + lunch = brunch
helicopter + airport = heliport
smoke + fog = smog
Channel + Tunnel = Chunnel
Yale + Harvard = Yarvard
slang + language = slanguage
guess + estimate = guesstimate

(See David Crystal "The Cambridge Encyclopedia of the English Language", p. 130)

Look through the text once again and find a blend referring to two famous places in Britain.

6. *Complete the sentences below with one word only. Find out the meaning of the italicized phrases.*

1) When I was a graduate student at Harvard, in the early 1950s, the question of whether anything that was taught or studied in the Mathematics Department had any practical applications could not even be asked, *let* ... discussed.

2) Today, 50 years after Hardy's death, it seems incredible that a book so *at ... with* reality was so influential for so many years.

3) In many areas, mathematics was *running so far ... of* applications that it was widely assumed that most of these fields would never have any.

4) Rather than be apologetic about the lack of applications for many years, leading mathematicians and mathematics departments decided *to turn this possible defect into a ...*

5) ... but I believe I have *made my ...* .

6) Two years later, I was systematically reading Landau's *Vorlesungen über Zahlentheorie*, still *on my ...* .

7) It was only when Brun explained his method to me that it *... sense.*

8) The Soviet Union's launch of Sputnik 1, on October 4, 1957, *took the world by ...* .

9) It was *visible to the ... eye* in the night sky, and a fairly simple radio receiver could pick up its "beep-beep" signal.

10) *It* all ... *down to* this.

7. *Explain the use of the articles in the following sentences:*

1) "*The* 'real' mathematics of *the* 'real' mathematicians, *the* mathematics of Fermat and Euler and Gauss and Riemann, is almost wholly 'useless'; and we have concluded that *the* trivial mathematics is, on the whole, useful, and that *the* 'real' mathematics, on the whole, is not."

How far do you agree with this statement? Why?

2) "Another famous professor at Göttingen during *the* Hilbert epoch was Felix Klein, who had *a* much broader appreciation of application."

3) "At *a* time when Latin and Greek were indispensable parts of *a* university education, no one would have remotely considered eliminating mathematics as 'impractical'. Those students seeking *a* liberal university education, whether at Oxbridge in *the* U.K. or in *the* Ivy League in *the* U.S., were not thought to be concerned with learning *a* trade and earning *a* living."

4) "*An* atom, *an* electron, *a* photon, or *a* quark – these are all entities in *the* physical world whose behavior *the* physicist attempts to model."

8. "According to a famous story, a reporter once asked Klein if it was true that there was a conflict between "pure" and "applied" mathematics. Klein replied that it was wrong to think of it as a conflict, that it was really a complementarity. Each contributed to the other. The reporter then went to Hilbert, and told him, "Klein says there's no conflict between pure and applied mathematics." "Yes," said Hilbert, "of course he's right. How could there possibly be a conflict? The two have absolutely nothing in common." Since Hilbert, unlike Hardy, did work in areas of mathematics with obvious applications, and if the quote is authentic rather than apocryphal, the fundamental distinction he may have seen between pure and applied mathematics would likely have involved motivation - do we study it because it is beautiful or because it is useful?"

Do you study mathematics because it is beautiful or because it is useful? Write an essay (approximately 250 words). Remember to use linking devices to make your piece of writing coherent.

9. “Culture can play an important role in determining which mathematical questions are asked, and which mathematical topics are studied. (Our widespread use of the decimal system is undoubtedly related to humans having ten fingers.)”

How far do you agree with the above statement? Write an essay of approximately 250 words. Give examples to prove your opinion.

10. Hardy asserts that mathematicians are attracted to the subject by its inner beauty, rather than by any overwhelming desire to benefit humanity. Most mathematicians I know would agree with that.

What is the inner beauty of mathematics? Write an essay (approximately 250 words).

11. *Read the quotations below. Answer the questions.*

Joel Henry Hildebrand (1881-1983), American chemist.

Chemistry is fun!

1. Do you agree with Hildebrand?

2. What made him think so?

3. Do you think that mathematics is fun? Why? Why not?

4. Can a scientist achieve anything serious with Hildebrand’s idea in mind?

Max Born (1882-1970), German physicist.

All attempts to adapt our ethical code to our situation in the technological age have failed.

Jacob Bronowski (1908-74), Polish-born British mathematician and science writer.

Science has nothing to be ashamed of, even in the ruins of Nagasaki.

John Burden Sanderson Haldane (1892-1964), British geneticist, physiologist and biochemist.

Man armed with science is like a baby with a box of matches.

Adlai Ewing Stevenson (1900-65), American politician.

Nature is neutral. Man has wrested from nature the power to make the world a desert or make the deserts bloom. There is no evil in the atom; only in men's souls.

1. Which of the scientists do you agree with?

2. Is there any contradiction among them?

3. Could you reconcile their ideas with each other?

4. Do you think Haldane's quotation is related to that of Born? And Stevenson's one to that of Bronowski?

5. Give examples justifying Haldane's words. (NB: he was a geneticist.)

Bhurrhus Frederic Skinner (1904-90), American psychologist.

The real question is not whether machines think but whether men do.

6. Can you outline the trend in the development of robots and machines in general?

7. What kind of robot would you create if you could?

8. Do you think there is any danger posed by the development of AI?

Richard Dawkins (1941-), English evolutionary biologist.

We are survival machines, robot vehicles blindly programmed to preserve the selfish molecules known as genes. This is a truth which still fills me with astonishment.

It is raining DNA outside.

DNA neither cares nor knows. DNA just is. And we dance to its music.

1. Try to explain the second quotation.

2. Do Dawkins's words eliminate the idea of the soul?

3. Can we say that evil things are inevitable as everything is determined by genes?

XI. The Planiverse Project: Then and Now

A.K. Dewdney, The Mathematical Intelligencer, vol. 22, number 1, Winter 2000, pp. 46-51.

Is a two-dimensional universe possible, at least in principle? What laws of physics might work in such a universe? Would life be possible? It was while pondering such imponderables one steamy summer afternoon in 1980 that I came to the sudden conclusion that, whether or not such a place exists, it would be possible to conduct a gedanken experiment on a grand scale. It was all a question of starting somewhat mathematically. With the right basic assumptions (which would function like axioms), what logical consequences might emerge?

Perhaps the heat was getting to me. I pictured my toy universe as a balloon with an infinitesimal (that is to say, zero-thickness) skin. Within this skin, a space like ours but with one dimension less, there might be planets and stars, but they would have to be disks of two-dimensional matter. In laying out the basic picture I followed informal principles of simplicity and similarity. Other things being equal, a feature in the planiverse should be as much like its counterpart in our universe as possible, but not at the cost of simplicity within the two-dimensional realm. The simplest two-dimensional analog of a solid sphere is a disk.

What sort of orbits would the planets follow? In our own

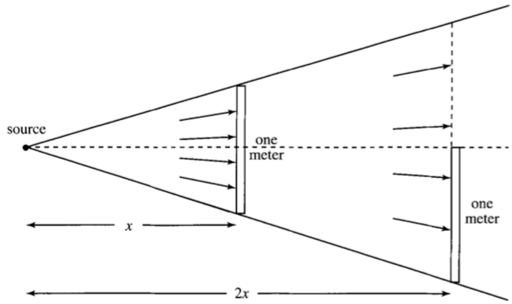

Figure 1. The law of gravity.

universe, Newtonian mechanics takes its particular form from the inverse-square law of attraction. A planet circling a star, for example, "feels" an attraction to that star which varies inversely with the square of the distance between the two objects. The same reason in the planiverse leads to a different conclusion. The amount of light that falls on a linear meter at a distance $2x$ from a star is one-half the light that reaches the square at a distance x from the star. (see Figure 1); correspondingly, attraction is proportional to the inverse first power of the distance.

The resulting trajectory is not a conic section, but a wildly weaving orbit, as in Figure 2.

The figure resembles a production of that well-known toy, the spirograph, in which gears laid on a sheet of paper roll around each other. A pencil inserted in a hole in one of the gears might trace such a figure. Are the two-dimensional orbits spirograph figures? Probably not. They look like epicycles, the paths that early astronomers thought might explain the looping orbits of Mars and Jupiter in an Earth-centered system! (It is tempting to conclude that what goes around comes around.)

Encouraged by such speculations, I begin to develop the impression that such a universe might actually exist. It would be completely invisible to us three-dimensional beings, wherever it might be. But places, even imaginary ones, need names. What could a two-dimensional universe be, but the Planiverse?

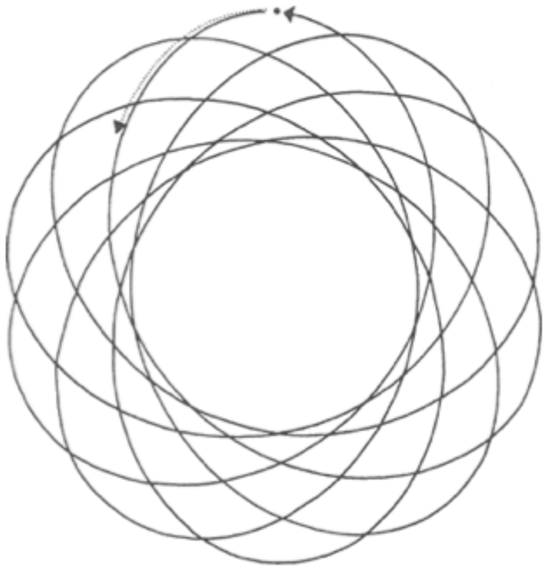

Figure 2. Orbit of a two-dimensional planet.

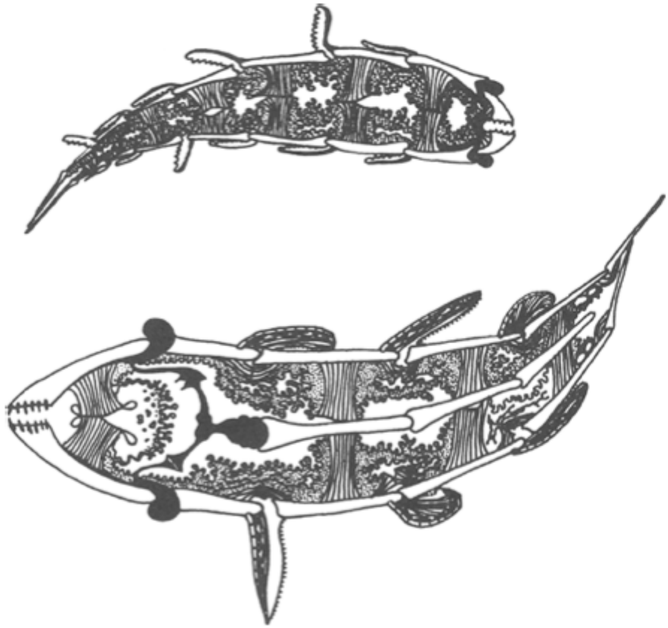

Figure 3. Two-dimensional fish.

Figure 4. Planiversal vehicle.

In a fit of scientific irresponsibility I sent a letter to Martin Gardner, then author of the Mathematical Games column for *Scientific American* magazine. I included several speculations, including the drawing of a two-dimensional fish shown in Figure Three below.

Gardner wrote back, saying that he not only found the planiverse a delightful place, he would devote a forthcoming column to it. His column, which appeared in July, 1980, lifted our speculations about two-dimensional science and technology to a new level by bringing it to the attention of a much wider public. Among those who read Gardner's column were not only scientists and technologists, but average readers with novel and startling contributions of their own.

I left for a sabbatical at Oxford that summer, hoping to work on the theory of computation and hoping also to get away from the planiverse project, which was claiming more and more of my time. I stayed in an abbey in the village of Wytham, near Oxford. There was leisure not only to work on the logical design for an entirely new way to compute things, but the opportunity to work on the Planiverse Project, a paper symposium with colleague Richard Lapidus, a physicist at the Stevens Institute of Technology in New Jersey. Our symposium had contributions from around the world on everything from two-dimensional chemistry and physics to planetary theory and cosmology. There was, moreover, a section devoted to technology, wherein the only feasible two-dimensional car ever designed appeared for the first time. It had no wheels, but was surrounded by something like a tank tread that ran on disk-bearings. The occupants got in and out of the vehicle by unhooking the tread.

The Planiverse Project was now proceeding at a satisfying rate. I assumed that within a few years it would die away to nothing. We would have had our fun, no harm done.

But a press release, written by a journalist at my home institution in the fall of 1981, changed all that. Wire services picked it up with the glee reserved for UFO reports and escaped lions. There followed a rush of magazine and newspaper articles, as well as television stories publicizing our two-dimensional world.

In particular, a piece in *Newsweek* magazine caught the attention of publishers.

In the midst of a series of papers on programming logic, I was suddenly face to face with a big writing job. There were contracts with Poseidon Press (Simon & Schuster) in the US, with Pan/Picador in England, and with McClelland & Stewart in Canada. I viewed these new responsibilities with irritation. It was assuredly fun to think about the planiverse, but my research came first. And was I not in danger of being regarded as a nut-case? The media were no help. One interviewer asked, "So, Professor Dewdney. Are you saying the Earth is flat after all?" (He was serious!)

The writing job, as I finally came to view it, would have to weave together all the scientific and technical elements that had emerged from the Planiverse Project. But a compendium of these speculations, no matter how wild or entertaining, would surely prove a dry read. It would have to be a work of fiction, set in the planiverse itself. There would be a planet called Arde, a disc of matter circling a star called Shems. There would be a hero named Yendred (almost my name backwards) and his quest for the third dimension or, at least, a spiritual version of it. Yendred is convinced that the answer to his quest lies on the high plateau of Arde's lone continent (a requirement of two-dimensional plate tectonics).

All the elements of our earlier speculations now fell more or less into place. Think for a moment of even the humblest respects in which a two-dimensional existence on the "surface" of Arde might differ from our own.

The Jordan curve theorem's implications for Arde were profound. Closed curves lurked everywhere.

Consider, for example, Ardean soft, a mechanical mixture of two-dimensional grains and pebbles in which any pocket of water finds itself permanently trapped within the closed circle of surrounding stones. The water cannot percolate, as our groundwater does, up or down. It is trapped, at least until the soil is mechanically disturbed. Consider also the simple matter of Yendred attempting to lift a two-dimensional plank on the Ardean surface. The plank, the ground, and Yendred himself would form a

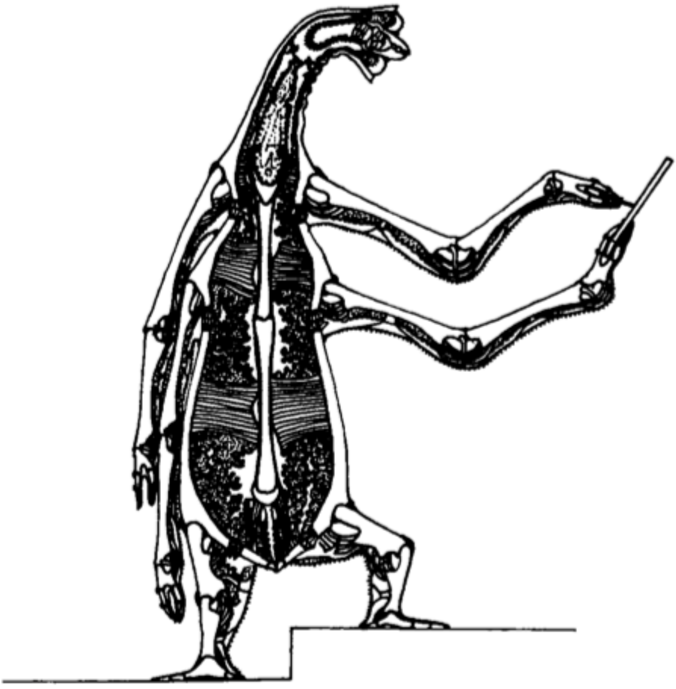

Figure 5. Yendred, a typical Ardean.

simple closed curve, and the air trapped inside the enclosed space would become increasingly rarefied. The plank would seem to get heavier and heavier. Perhaps readers can imagine themselves to be Ardeans lifting such a plank. If you were Yendred, what technique would you adopt to make it easier?

But for every disadvantage of life in two dimensions, there seems to be an equal and opposite advantage. Bags and balloons are trivial to make–from single pieces of string! Yendred's father, who takes him fishing near the beginning of the book, never has trouble with tangled lines, for knots in two-space are impossible. Moreover, sailing requires nothing more than a mast!

Yendred sets out on his quest shortly after the fishing trip with his father. His home, like all Ardean homes, is underground. The surface of Arde must be left as pristine as possible. There are travelling plants and periodic rains which make temporary rivers, basically floods. Any surface structure would either disrupt the delicate one-dimensional ecology or be swept away, in any case. A simple pole stuck in the ground would become a dam which could never withstand the force of kilometers of water that would rapidly build up behind it.

In the Ardean cities which Yendred must walk through (or over) on his travels to the high plateau, we encounter the acme of two-dimensional infrastructure. There is no skyline, only the typical Ardean surface periodically marred by traffic pits. If an eastbound Ardean should happen to encounter a westbound colleague, one of them must lie down and let the other walk over him/her. Elaborate rules of etiquette dictate who must lie down and who proceed, but in an urban context there is no time for niceties. Whenever a westbound group of Ardeans encounters a west-pit, they descend the stairs, hook up an overhead cable and walt. At the sound of a traffic gong, an eastbound group marches across the cable. What would be a tightrope act in our world amounts to little more than a springy walk in two dimensions for the eastbounders. West-pits and east-pits alternate so that neither direction has an advantage over the other.

From a privileged view outside the Planiverse, the "skyline" of an Ardean city resembles an inverted Earth-city skyline. Yendred

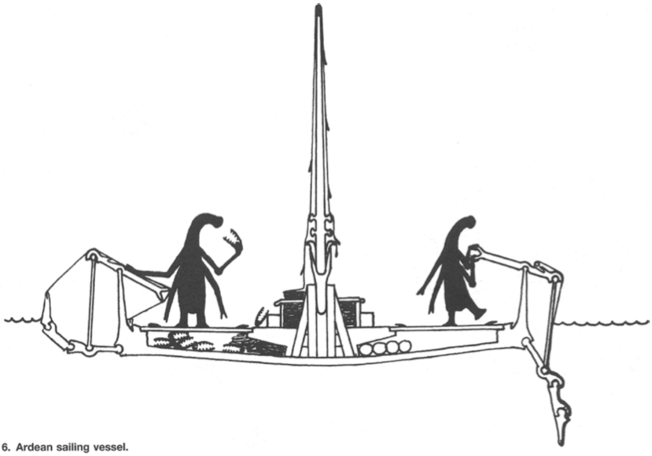

Figure 6. Ardean sailing vessel.

passes over numerous houses, apartment buildings, and factories, marked only by the exit or entrance of fellow citizens bent on private tasks like so many two-dimensional ants. Overhead pass delivery balloons, each with its cargo of packages. Balloon drivers adjust to near-neutral buoyancy, then take great hops over their fellows.

Access to underground structures is managed by swingstairs. Although some of the larger structural beams are held together by pegs, the fastener of choice is glue. Wires (yes, the Ardeans have electricity) run only short distances, from batteries to appliances. Electrical distribution is out of the question since power lines would trap everyone within their homes. Reading by the feeble glow of a battery-powered lamp, an Ardean might reach for his favorite book, reading text that resembles Morse Code, one line per page. This demands a highly concentrated prose style that is more suggestive than comprehensive.

The population of Arde is not great. Only a few thousand individuals inhabit its lone continent. Consequently, the Ardeans have no great demand for power machines, the steam engine sufficing for most needs, such as elevators and factories. Readers might be able to figure out the operation of an Ardean steam engine from the accompanying illustration alone.

A boiler converts water into steam, and when a valve opens at the top of the boiler, the steam drives a piston to the right. However, this very motion engages a series of cams that close the valve. The steam then enters a reservoir above the piston and escapes when the piston completes its travel to the head of the "cylinder." Interestingly, almost any two-dimensional machine can also be built in three dimensions. It must be given some thickness, of course, and it must also be enclosed between two parallel plates to simulate the restriction of no sideways movement. I have often wondered whether we could build a car with a one-inch thick steam engine mounted underneath. Think of the additional room that would provide!

Ardean technology is a strange mixture of advanced and primitive machines. Although steam engines are the main power source, rocket planes travel from city to city, while space satellites orbit overhead. It is absurdly easy to make space stations airtight. Any structure that contains at least one simple closed curve is automatically airtight.

And of course, there are computers! These operate on the same binary plinciples (0 and 1) as our own do. Ardean technologists had a difficult time developing the appropriate circuits, however, owing to the impossibility of getting wires to cross each other. One brilliant engineer finally hit on the idea of a "logic crossover." Symbolically rendered below, this circuit consists of three exclusive-or gates, each transmitting a logic 1 signal if and only if exactly one input is a 1.

No matter what combination of zeros or ones enter this circuit along the wires labelled x and y, the same signals leave the circuit along the wires bearing these labels. Readers may readily satisfy themselves that if x and y both carry a zero (or one), for example, then both output lines will also carry this signal. But if x is one and y is zero, the middle gate will output a one which will cancel the x-signal in the upper gate and combine with the zero on the y-input in the lower gate to produce a one.

Fun though technology may be, it isn't until he visits the Punizlan Institute of Technology (PIT) that Yendred encounters the deep scientific ideas of his time. Scientists at PIT have

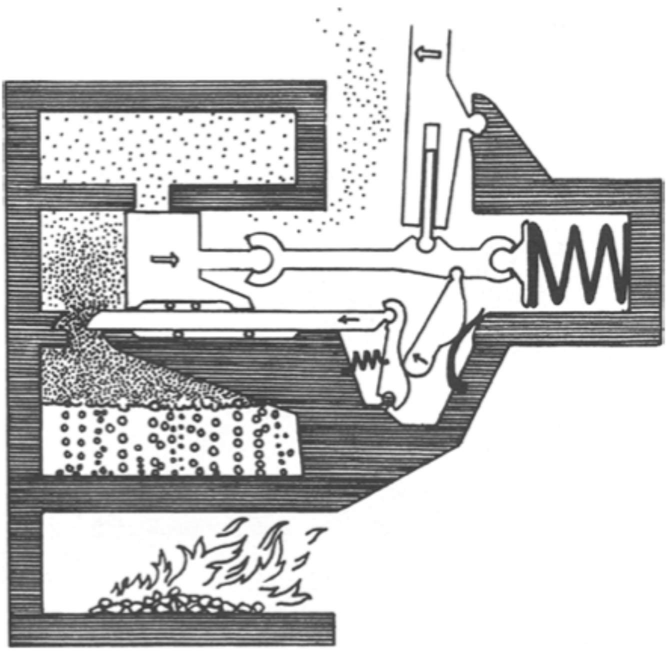

Figure 7. A steam engine.

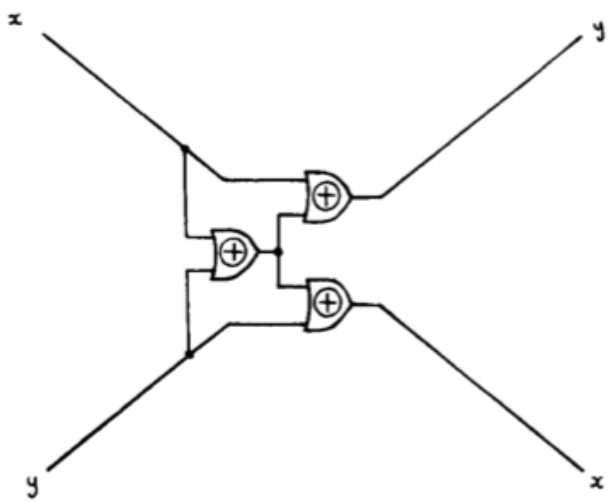

Figure 8. A logic crossover.

developed a periodic table of the elements based on the theory that while just two electrons can occupy the first shell of a planiversal atom, up to six can occupy the second shell. We have labelled the planiversal elements with the symbols of the elements from our own universe which they most resemble.

Strangely, the planiversal elements quickly run out, owing to the instability of very large planiversal atoms. In the planiverse, one simply cannot pack as many neutrons and protons into a small space as one can in our universe. Consequently, nuclear forces (other things being equal) must act across larger distances and the nuclear components are rather less tightly bound. Quite possibly, there is a lot more radioactivity in the planiverse than in our own.

Other strange features of the planiverse include rather low melting points and the strange behaviour of sound waves. Low melting points might militate against the possibility of life, except that chemical reactions proceed at lower temperatures, in any event. Sound waves travel much farther and have a very strange property first deduced by Earth scientists some time ago. If one

1 H													2 He
3 Li	4 Be									5 C	6 O	7 F	8 Ne
9 Na	10 Mg									11 Si	12 S	13 Cl	14 Ar
15 K	16 X					17 X	18 X	19 X	20 X	21 X	22 X	23 Br	24 Kr
25 Rb	26 X					27 X	28 X	29 X	30 X	31 X	32 X	33 I	34 Xe
35 Cs	36 X	37 X	38 X	39 X	40 X	41 X	42 X	43 X	44 X	45 X	46 X	47 At	48 Rn

Figure 9. Planiversal table of the elements.

sounds a note on Arde, the sound wave alters as it travels. A sharp attack smears out in time, so that a single note of C, for example, is heard at a distance as a glissando rising from some lower pitch and asymptotic to C.

Cosmologically speaking, Ardean scientists have much to ponder. Like us, they wonder if their universe is closed like a balloon (we say it is) or open like a saddle-shaped space. It is apparently expanding, and the balloon analogy, so often used to illustrate how our own universe is apparently expanding, can be taken quite literally. A deeper question concerns the orientability of the planiverse. Perhaps it is really a projective plane, so that Yendred, travelling by rocket across the planiverse, might return to find that everyone has reversed their handedness and all Ardean writing appears backward.

As for space travel, another problem awaits the rocket voyager. There is no escape velocity in the planiverse. The amount of work required to escape the gravitational field of an isolated planet is infinite! (Try integrating 1/x from 1 to infinity.) However, if one can travel far enough to fall under the gravitational influence of some other body, the infinite escape velocity no longer matters.

The Planiverse Project had the most fun designing two-dimensional life forms. Readers who turn back to the picture of the fish (Figure 3) will find a creature with a well-developed exoskeleton, like an insect, and with a rudimentary endoskeleton, as well. The key anatomical component in any two-dimensional life form is the zipper organ, two strips of interdigitating muscle that meet to form a seam. Just inside the fish's bony jaws, for example, the muscles which crush and chew the prey also part to admit its fragments into a digestive pouch. Because portions of the two muscles are always in contact, structural integrity is maintained. The fragments are enclosed in a pocket that travels along the seam from front to back.

Yendred, after many adventures, finally reaches the high plateau and meets the mysterious Drabk, an Ardean who has developed the ability to leave the planiverse entirely and move "alongside" it, so to speak. Since *The Planiverse* is about to re-appear, I will not give the plot away, but I had better mention the *deus ex machina* that makes it all possible: in the book a class project results in a program called 2DWORLD that simulates a two-dimensional world, including a disk-shaped planet the students call Astria. Imagine the student's surprise when 2DWORLD turns out to be a sophisticated communication device which, by a Theory of Lockstep, begins to transit images of an actual two-dimensional universe, including a planet called Arde and a being called Yendred!

When *The Planiverse* first appeared 16 years ago, it caught more than a few readers off guard. The line between willing suspension of disbelief and innocent acceptance, if it exists at all, is a thin one. There were those who wanted to believe (despite the tongue-in-cheek subtext) that we had actually made contact with a two-dimensional world called Arde.

It is tempting to imagine that those who believed, as well as those who suspended disbelief, did so because of the persuasive consistency in the cosmology and physics of this infinitesimally thin universe, and in its bizarre but oddly workable organisms. This was not just your run-of-the-mill science fiction universe fashioned out of the whole-cloth of wish-driven imagination. The planiverse is a weirder place than that precisely because so much of it was worked out in the Planiverse Project. Reality, even the pseudo-reality of such a place, is invariably stranger than anything we merely dream up.

Other Attempts at Two-Dimensional Universes

The Planiverse has had a long evolutionary history, marked by previous books on two-dimensional worlds. The first of these was *Flatland*, written in 1884 by Edwin A. Abbott, an English clergyman. Some years later, in 1907, Charles Hinton, an American logician, wrote *An Episode of Flatland*, which reorganised Abbott's tabletop world into the somewhat more logical disk planet that he called Astria. Much later, in 1965, Dionys Burger, a Dutch physicist, published *Sphereland*, which attempted to reconcile Abbott's and Hinton's worlds and then to use the resulting two-dimensional universe to illustrate the curvature of space.

For all their charm, these books have various shortcomings. Abbott made no attempt to endow his universe with coherent physics. His beings float about in two-space with no apparent mode of propulsion. Being geometrical figures, they have no biology at all. Hinton's universe is rather more like the planiverse, his planet being a disk. But Hinton, immersed in a sort of socialist Utopian fantasy, keeps forgetting the restrictions of his characters' two-dimensionality, seating his characters "side by side" at a banquet, for example. Berger attempts to reconcile the two previous universes, but he is really after just an expository vehicle to illustrate various ideas about space and physics.

Exercises

1. *Specify the italicized words below and translate the sentences into Russian:*

1) Is a two-dimensional universe possible, at least in principle? What laws of physics *might* work in such a universe? *Would* life *be* possible? It was while *pondering* such imponderables one steamy summer afternoon in 1980 that I came to the sudden conclusion that, whether or not such a place exists, it *would be* possible to conduct a gedanken experiment on a grand scale. It was all a question of *starting* somewhat mathematically. With the right basic assumptions (which *would function* like axioms), what logical consequences *might* emerge?

2) Within this skin, a space like ours but with one dimension less, there *might be* planets and stars, but they *would have* to be disks of two-dimensional matter.

3) *Other things being equal*, a feature in the planiverse *should be* as much like its counterpart in our universe as possible, but not

at the cost of simplicity within the two-dimensional realm.

4) The *resulting* trajectory is not a conic section, but a wildly *weaving* orbit, as in Figure 2.

5) *Encouraged* by such speculations, I begin to develop the impression that such a universe *might* actually exist. It *would be* completely invisible to us three-dimensional *beings*, wherever it *might* be. But places, even imaginary ones, need names. What *could* a two-dimensional universe be, but the Planiverse?

6) The occupants got in and out of the vehicle by *unhooking* the tread.

7) Consequently, the Ardeans have no great demand for power machines, *the steam engine sufficing for most needs*, such as elevators and factories.

8) However, this *very* motion engages a series of cams that close the valve.

9) Consequently, nuclear forces (*other things being equal*) must act across larger distances and the nuclear components are rather less tightly bound.

2. *What law of physics does the following sentence remind you of?*

"But for every disadvantage of life in two dimensions, there seems to be an equal and opposite advantage."

3. *Give the Russian equivalents to the following English terms. Use your English-English dictionary to check the pronunciation.*

Dimension, realm, a solid sphere, a disk, infinitesimal, Newtonian mechanics, the inverse-square law of attraction, a figure, distance, a trajectory, a conic section, a closed curve/circle, a knot, two-space, buoyancy, a melting point, a sound wave, a projective plane, escape velocity, a gravitational field, infinite/finite.

4. *Add the missing letters to complete the words below. Use your dictionary to check your answers, if necessary.*

Two-dimen...ional, univer...e, ph...sics, exp...r...ment, re...lm, prin...iple, simpli...ity, mec...anics, inver...e, squ...re, re...emble, fig...re, sab...atical, ent...rely, colle...gue, fe...sible, plan...tary, perman...ntly, s...rrounding, pr...ceed, s...ffice, bin...ry, feat...re, as...mptotic, accept...nce, suspen...ion.

5. *What are the plural forms of the following nouns? Use your dictionary, if necessary.*

Compendium, formula, medium, symposium, nucleus, radius.

6. *Find the following phrasal verbs in the text and try to guess their meaning.*

lay out
give away
turn out
run out
figure out
sweep away
set out
get away

Now use them to complete the sentences below:

1.We ... ... to conduct the experiment as soon as we got the material. 2. He ... ... his papers and explained the conditions of the agreement. 3. We are ... ... of time. 4. Once you have made an outstanding contribution, it is next to impossible to ... ... from public attention. 5. It is easy to ... ... the results of the experiment. 6. While working on the proof of Fermat's Last theorem A.Wiles did not of ... ... his secret to the colleagues. 7. Imaginary numbers ... ... to be extremely useful in the solving of the equations that could not be solved previously. 8 The tsunami ... ... whole cities lying on the coast. 9. There's no ... ... from it.

7. *Explain the use of the articles in the following sentences:*

1) Within this skin, *a* space like ours but with one dimension less, there might be planets and stars, but they would have to be disks of two-dimensional matter. 2) Other things being equal, *a* feature in *the* planiverse should be as much like its counterpart in our universe as possible, but not at the cost of simplicity within

the two-dimensional realm. 3) A planet circling a star, for example, "feels" *an* attraction to that star which varies inversely with the square of *the* distance between *the* two objects. The same reason in the planiverse leads to *a* different conclusion. *The* amount of light that falls on *a* linear meter at *a* distance 2x from *a* star is one-half *the* light that reaches *the* square at *a* distance x from *the* star; correspondingly, attraction is proportional to the inverse first power of the distance.

8. *Read through the article once again and choose the word combinations and phrases that may help you describe the geometric and mechanical properties of the author's planiverse. Now write a summary of the article outlining the difference between the geometric and mechanical properties of the planiverse and those of our universe. Remember to use linking devices to make your summary coherent.*

NB! You have come across a most useful linking phrase in the article, namely *cosmologically speaking*. It can be *mathematically/ geometrically/ ... speaking*. Try and use it in your own piece of writing.

9. *Read the quotations below. Answer the questions.*

John Desmond Bernal (1901-71), Irish physicist and x-ray crystallographer.

The beauty of life is, therefore, geometrical beauty of a type that Plato would have much appreciated.

1. Is nature beautiful in the sense Bernal understood it?

Godfrey Harold Hardy (1877-1947), British mathematician.

The mathematician's pattern, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics.

2. Could you give examples of ugly mathematics? Beautiful mathematics?

3. Is it really necessary for mathematics to be beautiful?

XII. The Foundations of Geometry and the History of Geometry

Jeremy Gray, The Mathematical Intelligencer, Vol. 20, Number 2, Spring 1998, p. 54.

The Foundations of Geometry and the History of Geometry

When historians of mathematics seek to explore the work of another mathematical culture, they naturally draw on their own experience and that of the mathematicians around them. This approach brings insights no careful reproduction of the texts can manage, but it brings the risk of misreading too. The celebrated example of non-Euclidean geometry, one of the most actively discussed topics in 19th-century mathematics, is a case in point. *Indeed, the early years of the axiomatisation of geometry produced a philosophy of geometry that had a marked effect on the writing of the history of mathematics at the time.*

It is usual to suggest that Hillbert's Grundlagen der Geometrie, first edition 1899, marks the start of the move towards axiomatising mathematics, or to be more precise, towards giving formal, "meaningless" axiom systems as the basis of each mathematical discipline and eventually of all mathematics (see Kline, or Toepell, who also quotes Hurwitz to this effect in 1903). This enterprise, which is also taken to expire with Gödel's work in the 1930s, is often referred to as Hilbert's formalist programme. The bulk of Hilbert's work on it, a few forays aside, is concentrated in the 1920s and deals with difficult questions about the relation of mathematics to logic. On this view, the importance of the work on the foundations of geometry is that it prefigures the later work, and, it is sometimes suggested, inspired similar treatments of other topics – group theory is often mentioned, along with Zermelo's axiomatization of set theory (see Moore) and Steinitz's theory of fields. It is often observed that the Grundlagen der Geometrie ran to 12 editions, changing its nature considerably over the years as several appendices were added, and that it was translated into several languages – proof of the esteem in which it was held.

There are a number of problems with this view. The first edition does not fit the bill very comfortably: it is not about Euclidean geometry but about various non-Archimedean geometries, presented in the form of five families of axioms with particular attention to the question of what underlying continuum is presupposed. Only in the second edition are non-Euclidean geometry and the independence of the parallel postulate discussed. There are problems with the quality of the argument all the way up to the 8th edition, the first to be published after Hilbert's death in 1943. The edition that was translated into English was the second; the better-known second English translation corresponds to the 10th German edition. The source of the most rigorous critiques of Hilbert's work was Freudenthal, who also wrote one of the few good historical accounts of it, and it is with the history that I shall be most concerned.

I shall take history here in two senses: I want to look, first, at the early years of the axiomatisations of geometry; second, at the effect this philosophy of geometry had on the writing of the history of mathematics at the time. These are connected, because the historians of mathematics (and it was a golden age for the history of mathematics) were professional mathematicians working in good mathematics departments.

The awkward fact, which Freudenthal pointed out, is that around 1900 the axiomatisation of geometry is more an Italian than a Germany story; Kline's account concurs. The story does start in Germany: Moritz Pasch's Vorlesungen uber neuere Geometrie is the first book in which a thorough reworking of geometry is proposed. *He sought to formulate rigorously every fact about plane projective geometry, starting with undefined or primitive concept of the straight line segment between two points.* Results or properties about segments he felt necessary to assume without proof he called Grundsatze. All Grundsatze were, he said immediately grounded in observation, and he cited Helmholtz's paper "On the origin and significance of the geometrical axioms" at this point. Results he could deduce from the Grundsatze he called Lehrsatze. There were 8 Grundsatze needed to base the theory of line segments, of which the first is "there is always a unique segment joining any two points." In general, Grundsatze should be laid down until the mathematician could henceforth reason logically and without further appeal to sense perceptions. The rest of the book is devoted to showing that that can be done.

Thereafter, the study of geometry from an axiomatic point of view was taken up most eagerly in Italy. One of the tricky topics, and one that most interested Hilbert, was the interdependence of the Theorems. One source for this interest was in the delicate business of sorting out projective geometry. A point D is called the harmonic conjugate of B with respect to A and C if the 4 points are collinear and a complete quadrilateral can be found such that 2 pairs of opposite sides pass through A and C and two diagonals through B and D. Desargues's Theorem was then used to prove the uniqueness of the 4th harmonic point.

The fundamental Theorems of projective geometry are that a projective correspondence between two lines is completely determined when three distinct points on the one are mapped onto the other, and that any correspondence between two lines that preserves harmonic conjugates is a projective correspondence. Similar statements about four points in the plane also hold.

Christian Wiener had been the first to note that the proof of the first fundamental Theorem relies on the theorems of Desargues and Pappus, but he did not prove it. The first to do so was Schur

[1889]. After this, Hilbert had shown in his Grundlagen, Chapter 6 that indeed the first 15 axioms do not imply Pappus's Theorem. As Desargues himself had observed, the proof of Desargues's Theorem is automatic in projective spaces of dimensions 3 (or more), but in two dimensions it requires a special proof. Peano in his [1894] gave a proof in the three-dimensional case, but refrained from comment on the situation in two dimensions, which left the matter unresolved (as van der Waerden's [1986] confirms). Then Hilbert in his Grundlagen [1899], with a further simplification by Vahlen [1905] and most simply Moulton [1902], noted that one can have a projective space satisfying the first 12 axioms and the contradiction of Desargues's Theorem may be false in the plane, a rather shocking result! Even the notion of harmonic conjugate is not as simple as it may seem. In his [1891] Fano noted, using the example of the finite projective plane with 7 points and 7 lines, that there need not be a harmonic conjugate at all, and using the finite space with 15 points he noted that the existence of a harmonic conjugate does not imply that A and C separate B and D.

Building on this series of discoveries of novel, counter-intuitive theorems in projective geometry, Italian mathematicians did not constrain their projective geometry to the facts of everyday experience. The significant novelty in Mario Pieri's work, which marks it out from Pasch's, is the complete abandonment of any intention to formalize what is given in experience. Instead, as he wrote in [1895], he treated projective geometry "in a purely deductive and abstract manner,..., independent of any physical interpretation of the premises." Primitive terms, such as line segments, "can be given any significance whatever, provided that they are in harmony with the postulates which will be successively introduced." In Pieri's presentation of plane projective geometry (Pieri [1899] nineteen axioms were put forward (typically: any two lines meet).

It was the Italian work rather than that of Hilbert which travelled best, to the English-speaking world at least, as the citations in A.N. Whitehead's Cambidge Tract The axioms of Projective Geometry (CUP Tract nr 4, 1906) show. One recalls that it was Peano's example that inspired Russel to take up mathematical logic. Whitehead's axiomatisation, citing the literature just described, used 12 axioms to describe the projective plane. Axiom 13 allowed for a point outside the plane. His treatment of order properties followed Pieri and his friend Bertrand Russell's Principles of Mathematics, Ch 24, 25. Then Whitehead introduced Fano's axiom, and soon had a system of 15, of which the last confined attention to 3-dimensional spaces. A further four axioms allowed the introduction of coordinates. Pieri's ideas spread to France, where they were summarised by Couturat in his Principles des mathematiques [1905] - a work openly inspired by Russell's work of the same name. *In America Pieri's system of axioms for geometry was adopted by the mathematician J.W. Young, who learned them from Couturat.* In his book [1911] he compared the systems of Hilbert, based on congruence, and Pieri, based on motion (in the sense of 1-1 transformation); finding the concept of congruence derived from it. Meanwhile, Veblen had come independently to some of these ideas. Whitehead particularly acknowledged the mimeographed notes of Veblen's Princeton lectures "On the foundations of geometry" published by the University of Chicago in 1905. The later two-volume work by Veblen and J.W. Young is even-handed in its attributions to Hilbert and to the Italians. *So it seems that in the early years of the 20th century Italian ideas, such as Pieri's, met with a greater degree of acceptance than is commonly recognised today.*

The question then arises why the Italians have been so forgotten. The obvious answer is part of the correct one. Hilbert was Hilbert: a powerful mathematician at the centre of the leading mathematical department in the world. However transient his interest in the foundations of geometry (and he had at most 4 PhD students in this area, out of a total of 60 in his life), he lent the subject his potent name. Thence the numerous editions and the translations. Add that he survived the War, and in the 1920s took up the foundations of mathematics, whereas Pieri died in 1913. Add that he was a lucid writer, whereas Pieri wrote in the unnaturally constricted style of his mentor, Peano. In this argot, as many words as possible of a natural language were eliminated in favour of symbols. Points of emphasis, overviews of the argument, motivation - all could be suppressed. Peano would sometimes write a paper in two halves, the first in this logical manner, the second in Italian prose. My subjective experience with such papers is that they are no easier or harder to read than others, but a lot less exciting.

After the war Peano lectured more and more in this style, to the distress of a Dean of the Faculty who came to investigate. It is surely obvious which style will make friends. So marked was the collapse of the Italian axiomatic geometers that Freudenthal calls their triumph a Pyrrhic victory. For a discussion of the strengths and weaknesses in logical terms of Peanian, as his mathematical language was colloquially called, the reader is referred to Segre's essay and Zaitsev.

Axiomatising elementary geometry is a process that comes to an end; it is, in more senses than one, a finite task, unless one is to plunge into deep questions of mathematical logic. The Italians pursued it as part of Peano's programme to write mathematics in completely unambiguous, logical fashion. Hilbert's taste was much broader, and that is also important. The Appendices to the Grundlagen der Geometrie include his famous theorem that there is no smooth isometric embedding of non-Euclidean two-dimensional space into R3 . It does not belong to axiomatics, but it does belong to mainstream mathematics, and I think that mathematicians found that reassuring once the chill wind of axioms had had its salutary effect. The rise of Gottingen, the diaspora after the Nazis, and the contemporary decline of Italian geometry into a presumed never-never land of hazy intuitions all preserved one tradition, and covered up another. The folk memory was established - as history.

On one interesting matter the folk memory is strangely silent. Just how much did Hilbert know of contemporary Italian work? In his thorough study of Hilbert's route to his Grundlagen (Toepell [1986]) Toepell shows that although Hilbert did not read Italian easily, he listed Peano's book on the Grassmann calculus in its German translation (Peano [1891]) as one of the books on the axioms of geometry. Toepell therefore disagrees with Morris Kline's remark that Hilbert "did not know the work of the Italians" (quoted from Kline [1972]). One might add that Italian geometers visited

Gottingen, where Hilbert became a professor in 1895. Significantly, Hilbert did not refer to Peano's much more axiomatic work, Peano [1889]. *But it would be hard to argue that Hilbert could have remained ignorant of Italian work after 1900.* The International Congress of Mathematics in Paris that year not only carried two papers by Alessandro Padoa on foundational questions, one was indeed devoted to expounding a new system of definitions for Euclidean geometry (Padoa [1902]). Padoa there made reference to much of the earlier Italian work, including Pieri's. Moreover, the Congress of Mathematicians came straight after the International Congress of Philosophers, at which many papers were presented on geometry, and several of the speakers attended both Congresses. *It is hard to imagine that Hilbert would not have been drawn into conversation on the topic.* He and the Italians may have begun separately, but their interests now flowed together. A measure of the degree to which Italian developments were not read may be Poincare's ignorance of them, to which Freudenthal drew attention (Freudenthal [1962]).

Two reasons for the disparity between Hilbert and the Italians may be worth nothing. When Hilbert spoke at the International Congress of Mathematicians in Paris, he referred to signs as memory aids. "Geometrical figures," he said, "are signs or mnemonic symbols of space intuition and are used as such by all mathematicians." Knowingly or not, this remark excluded Peano and those like him, for whom signs were, precisely, formal symbols. Then Hilbert went on to speak of the necessity of giving a rigorous axiomatic investigation of the conceptual content of geometrical signs and their combinations. *This is a hint to the second difference for Hilbert, the study of geometry in this new fashion led to different kinds of arithmetic based on the addition or multiplication of segments, not all of them equivalent to ordinary arithmetic, the segment arithmetic for Cartesian geometry.* This novelty held out the prospect of illuminating geometry over any algebraic number field, which was one of his prime purposes in pursuing all this research. Only by compartmentilizing Hilbert's work for the modern reader can he be represented as so thorough-going a formalist; the palm for that belongs to Italian mathemticians.

I. The History of Geometry

In an important paper of 1939, Ernest Nagel argued that the reformulation of geometry was an important source of modern logic. The kernel of his insight was that while duality in projective geometry puts points and lines in the plane on an exactly equal footing, intuition must prefer points. So mathematicians were forced away from intuition as the basis of geometry, and towards formalism and thence logic. I would add that non-Euclidean geometry promoted this tendency. It is the geometry that raises the question of the nature of space, and with it the embarrassing problem of explaining why mathematicians had been so wrong about geometry for so long. Nagel's example of the formalist geometer was, of course, Hilbert - but in this he was unfair, in ways that affect our understanding of Enriques. Indeed, as we shall see, the insight of Nagel owes a log to the original work of Enriques.

If Enriques was the spokesman internationally for Italian geometers, the one who most securely grasped the historical task a former pupil of his, Roberto Bonola. His La geometria non-Euclidea, which was published in 1906, grew out of an earlier essay written for a collection of monographs on geometry that Enriques had edited (Questioni riguardanti la geometria elementare, Bologna, Zanichelli, 1900). It was translated into German, and then into English by H.S. Carslaw, published by Open Court in 1912, where it appeared with a short Introduction by Enriques. The melancholy occasion of the preface was Bonola's death in 1911, at the age of 37. To produce the book, which is a classic still worth reading and has even been used as the basis of a course in geometry at Warwick (which is indirectly how I came to history of mathematics, and so to be a student of David Fowler's), Bonola relied very sensibly on the work of the indefatigable Friedrich Engel and Paul Stackel, who did so much to publish and publicise the work of Bolyai and Lobachevskii. Naturally Bonola, following the lead of Beltrami and Segre, also played up the significance of Gerolamo Saccheri, the Italian mathematician who had died in 1733 after discovering many results that now form the cornerstone of elementary non-Euclidean geometry. His work had lapsed into almost complete obscurity, and Beltrami had recently trumpeted its merits upon rediscovering it in 1889.

It was Engel, a former collaborator of Sophus Lie, and Stackel, who also worked on editing the extensive Gauss Nachlass under the direction of Felix Klein, who established the canonical version of the story of non-Euclidean geometry, through their editions of work by Lambert, Schweikart, and Taurinus as well as Bolyai and Lobachevskii. Their book, Theorie der Parallellinien von Euklid bis auf Gauss, 1895, is a generous collection of works by Wallis, Saccheri, and the later writers through to Gauss, to which they supplied a pertinent commentary. Engel himself translated Lobachevskii's two early and extensive Russian papers into German (Lobachevskii [1899]), and they tried to find as much as they could about the elusive figure of Janos Bolyai. Subsequent writers have discovered a great many minor figures omitted by them, but with the arguable exception of Legendre, no Western mathematician has entered the Pantheon. *One may conjecture that the reason is the progressive interpretation that can be placed on all this work, each author marking a significant advance until, while Schweikart hesitates and Taurinus looks backwards, Gauss takes the bold step into the non-Euclidean world.* From such a perspective, Legendre's attempts are reactionary, and sometimes embarrassingly flawed.

What did Engel and Stackel have to say? The range of material they presented is impressive; 19th-century German mathematicians were well-educated scholars. The names fly by, however, as so many precursors of the chosen few. Commentary amounts to 15-20% of the book, the rest being biographical and bibliographical. The thesis, if there is one, is concealed in the choice of authors.

Bonola's work, by contrast, makes more of an argument. With the Theorie der Parallellinien in print, he could content himself with summaries of the original sources, and tell a historical story of his own. He added a number of protagonists, expanded on the cryptic references to Arabic writers, included Legendre, and went on past Bolyai and Lobachevskii to Riemann. The first of five appendices considered the connection between the parallel postulate and the law of the lever; other appendices considered such topics as the independence of projective geometry from the parallel postulate, and the impossibility of proving the parallel postulate.

The generic account in Bonola's book of a mathematician's contribution goes like this. The mathematician's definition of a straight line and of a parallel line is given, or, if none was supplied, one is uncovered from the use of the concepts. The original argument is then presented in something close to its own terms, and the fallacy, if any, is explained. *So, when an argument that would seem to show that spherical geometry cannot exist is under discussion, Bonola shows how the postulate of Archimedes or its consequence, the indefinite extendibility of the straight line, has been tacitly invoked.* And when Legendre produces a fallacious argument using the postulate of Archimedes, Bonola also shows how it could have been avoided, the better to explain that this was not where Legendre erred.

From first to last, Bonola's account of the origin and development of non-Euclidean geometry is rooted in an analysis of axioms: their equivalence and their independence. Indeed, an Italian geometer, and a pupil of Enriques, writing between 1900 and 1911, would naturally see geometry as organised in this way. *It may be significant that Engel and Stackel were, if anything, more in the orbit of Klein than Hilbert, and being 40 in 1901-2, had several years of mature work behind them as differential geometers and analysts.* At all events, Bonola's work is analytic where theirs is descriptive. For Bonola, geometry is a matter of axioms, so the history is a history of axioms.

It need not have been so. The history of non-Euclidean geometry is open to other interpretations. *Had it just been a question of exhibiting an axiom system for something fairly geometrical, then spherical geometry would have done.* One needs, of course, to strike out two of Euclid's axioms: the parallel postulate and the indefinite extendibility of the straight line. That this was not done suggests that the ancients were not simply investigating axiom systems. It suggests, what a considerable amount of other evidence also suggests, that they were investigating something else: the geometry of physical space. The ongoing question was not "is the parallel postulate independent of the other axioms of geometry?", but "is the parallel postulate independent of the other axioms of geometry when giving an account of space?". This is a different enterprise from the much more overtly logical one in fashion around 1900.

There is another problem with Bonola's analytic approach: it is insensitive to the methods originally used. More precisely, the old arguments are presented carefully and accurately, but their significance is ignored. The book takes a dramatic turn on page 76 (in its English edition), with the first appearance of an analytic formula. Thereafter, the whole flavour switches from Euclidean-style arguments about angles and lines to hyperbolic trigonometry. We have entered a Chapter called "The Founders of Non-Euclidean Geometry," the early work of Gauss is behind us, and that of Schweikart and Taurinus is upon us. It rapidly becomes clear that this new trigonometry is the vital ingredient that made the discovery possible. Indeed, it is well-known that the work of Bolyai and Lobachevskii falls short of carrying logical conviction: it is a coherent description, but based upon an assumption about lines that was not rigorously defended. Their accounts are full of a vivid analogy between hyperbolic and spherical trigonometry. Elsewhere, in the early papers translated by Engel, Lobachevskii gave his own way of deducing the trigonometry from an analysis of geometry, but that is missing from Bonola's account. It was first clearly supplied by Beltrami, in his "Saggio" of 1868. But if hyperbolic trigonometry is the vital ingredient, one might ask who discovered it. The answer, as Bonola said in a footnote on page 82, is Lambert. This should have caught Bonola's attention, but he ducked the issue, caught as he was his axiomatic paradigm.

If mathematicians from before Euclid to-shall we say - Poincare were trying to describe space, then it would be natural for them to use trigonometry or conventional ("Euclidean") geometry. Axioms would appear, as they usually do in geometry before the advent of modern logic, as the undeniable truths they were taken to be, mixed up with definitions, stated explicitly or implicitly as is the way with statements of the obvious. Seen from this angle, which was obscured around 1900, the question as issue was not the logical status of the parallel postulate but something far more urgent: its truth. Paradoxically, the new-found clarity 100 years ago about the nature of mathematical reasoning perhaps led Bonola to emphasise the logic of the original arguments at the expense of their purpose.

II. End Note

There is much more to be said about Hilbert and axiomatisation. In forth-coming 'Years Ago' column, Leo Corry examines Hilbert's axiomatisation of physics, specifically, radiation theory.

Exercises

1. Translate the italicized parts of the text. Pay attention to the grammatical constructions.

2. Give the Russian equivalents of the following expressions:

to seek to explore; it is usual to suggest that; to be more precise; to deal with; with particular attention to the question; to correspond to; an awkward fact; to point out that; to lay down; to be devoted to; a conjugate harmonic function; with respect to; a counter-intuitive theorem; at least; the introduction of coordinates; in this argot; to be referred to; to plunge into deep question; to come straight after; to draw attention; to go on to speak; the conceptual content; to force away; an embarrassing problem; to lapse into obscurity; to establish the canonical version; to supply pertinent commentary; the indefinite extendibility of the straight line; a fallacious argument; to strike out; the ongoing question; more precisely; a coherent description; the undeniable truth.

3. Translate the following:

a) **Аксиома Паша.** Пусть A,B,C — три точки, не лежащие на одной прямой, и a — прямая в плоскости ABC этих трех точек, не проходящая ни через одну из точек A,B,C; если при этом прямая проходит через одну из точек отрезка AB, то она должна пройти через одну из точек отрезка AC или через одну из точек отрезка BC.

b) **Дедукция** — метод мышления, при котором частное положение логическим путём выводится из общего по правилам логики путем цепи умозаключений (рассуждений), звенья которой (высказывания) связаны отношением логического следования. Началом (посылками) дедукции являются аксиомы или просто гипотезы, имеющие характер общих утверждений ("общее"), а концом — следствия из посылок, теоремы ("частное"). Если посылки дедукции истинны, то истинны и её следствия. Дедукция — основное средство доказательства. Противоположно индукции. Пример дедуктивного умозаключения: Все люди смертны. Сократ — человек. Следовательно, Сократ смертен.

с) Два ненулевых (не равных 0) вектора называются **коллинеарными**, если они лежат на параллельных прямых или на одной прямой.

4. Translate the following from Russian into English using the word combinations from the text:

1. Изучение истории математики непрофессионалом в ряде случаев приводит к совершенно новым взглядам математиков на традиционные для них основы этой науки.

2. Одной из наиболее обсуждаемых математиками тем в XIX столетии была неевклидова геометрия.

3. Началом развития аксиоматики геометрии послужил философский анализ истории ее развития.

4. Начальный этап перехода к аксиоматизации геометрии обычно связывают с работой Гильберта "Основы геометрии", первый выпуск которой вышел в 1899 году.

5. Очень интересно понять, какое влияние работы Гильберта по аксиоматизации геометрии оказали на понимание истории развития геометрии и математики вообще.

6. Первой книгой, в которой было предложено полностью изменить взгляд на геометрию, можно считать «Лекции по новой геометрии» немецкого математика Морица Паша.

7. Когда Мориц Паш приступил к строгому определению всех основных положений проективной геометрии на плоскости, он начал с понятия отрезка между двумя точками.

8. Теорема Дезарга является одной из основных теорем проективной геометрии. Она формулируется так: если два треугольника расположены на плоскости таким образом, что

прямые, соединяющие соответствующие вершины треугольников, проходят через одну точку, то три точки, в которых пересекаются продолжения трёх пар соответствующих сторон треугольников, лежат на одной прямой.

9. По теореме Дезарга верно также и обратное: если два треугольника расположены на плоскости таким образом, что три точки, в которых пересекаются продолжения трёх пар соответствующих сторон треугольников, лежат на одной прямой, то прямые, соединяющие соответствующие вершины треугольников, проходят через одну точку.

10. В своих ранних работах, переведенных на немецкий язык Энгелем, Лобачевский разработал собственный метод дедукции тригонометрии на основе геометрии.

5. Read the text “History of Geometry” again and write a summary.

6. Read the quotations below and answer the questions.

Albert Einstein (1879-1955), German-born American physicist.

As far as the laws of mathematics refer to reality, they are not certain, they do not refer to reality.

1. On what grounds did Einstein make such a statement?

2. Give examples supporting his veiw.

3. Do you think there is any connection between these words and the ideas of Gödel and those of Kant and any other philosophers?

Plato (427-347/8 BC), Greek philosopher.

Let no-one ignorant of geometry enter. (Said to have been inscribed above the door of Plato’s Academy.)

4. Do you think Plato’s geometry was basically the same as modern geometry?

5. Could you say what role geometry played in Plato’s philosophy?

XIII. Geometry Problems Revisited

Alexander Shen, The Mathematical Intelligencer, Vol. 20, Number 2, Spring 1998, pp. 36-40.

This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate. We welcome, encourage, and frequently publish contributions from readers–either new notes, or replies to past columns.

After publishing a column about two- and three- dimensional geometric problems I got several letters about omissions in the column. For example:

To The Editor: The Mathematical Entertainments section (in Math. Intelligencer vol. 19, no. 3) tried to prove geometrically that if we take three intersecting circles, the chords joining pairwise intersections have a point in common. This is not quite true: the three lines are always in a pencil, but they may be parallel rather than concurrent. The intersecting circles

$$x^2 + y^2 + x + 5y = 0$$
$$x^2 + y^2 + 2x + 5y = 0$$

$$x^2 + y^2 + 5y + 2 = 0,$$

for instance, have all three chords parallel to the y-axis.

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I have to confess that that is not the only inaccuracy in the column; a lot of details are omitted. One omission is of special interest. In the chord problem it is possible that chords do not intersect at all, only the lines to which they belong do intersect:

The proof I gave does not work for this case. Indeed, it considered three spheres that have the given circles as diameter sections, and took the point in three-dimensional space that belongs to all three spheres. And now there is no such point.

However, there is a simple and rather general argument that allows us to extend the result to this case almost free. Let us consider the coordinates of the center points and the radii as variables. Then the coefficients of the chords' equations are functions of these variables, and the claim (three lines intersect in one point) becomes an equality. What functions are involved in this equality? It is clear that they are analytic (in fact, algebraic) functions. *So if the equality is true on a set of configurations that has a non-empty interior, the analytic continuation principle guarantees that it is valid everywhere. And, of course, the set of configurations with three intersecting circles has a non-empty interior, so we are done.*

The same remark can be applied to many other geometric problems. As Vahe Y. Avedissian points out in his letter, the situation arises in another problem mentioned in the same column: about the triangle and perpendiculars. The proof used a tetrahedron which does not exist in all cases. *But again the set of configurations where such a tetrahedron exists has a non- empty interior and the claim is an equality between analytic functions, so we can extend the result to all cases.*

However, some caution is needed when we use this argument: we should carefully check that all the functions are indeed analytic. For example, the distance between a point and a line is not analytic when the point crosses the line; it is only an absolute value of an analytic function. This is why we often need to consider oriented lengths and angles to make a general statement that is true for all configurations.

I. From a Special Case to a General Case

This way of reasoning (first prove something for a special case and then use some meta-argument to extend the result to the general case) is not limited to geometry. Here are some other nice examples.

II. Hamilton-Cayley Theorem

Let A be a linear operator in n-dimensional linear space. Let $P(\lambda) = det(A-\lambda)$ be its characteristic polynomial. Then $P(A) = 0$.

To see why it is true (for matrices over $\mathbb{C}$), we can use the following argument. First consider the case when all the eigenvalues of A are distinct complex numbers $\lambda_1, \ldots, \lambda_n$. Then the operator A has an eigenbasis where the matrix is diagonal with $\lambda_1, \ldots, \lambda_n$ on the diagonal. In this basis the equation $P(A) = (A - \lambda_1) \ldots (A - \lambda_n) = 0$ is evident, since the factors annihilate the coordinates one by one.

Now we have to extend this result to the general case. Let us consider elements of the matrix A (assuming some basis is fixed) as variables. Then the coefficients of P are polynomials in these variables, as are the entries of $P(A)$. Therefore, the claim is just an equality saying that some polynomial is equal to zero (actually we have n^2 equalities, not one). *Since polynomials are continuous functions and operators with distinct eigenvalues form a dense set, this equality must be true for all operators.* Q.e.d.

We can even make one step more and prove this theorem for any field. Indeed, these n^2 polynomials that form the equalities have integer coefficients, and these coefficients do not depend on the ground field. *So if the equalities are true for complex numbers,* *it just means that coefficients are all zero, so the statement is true for any field (or ring).*

III. Quantifier Elimination

The term "meta-argument" reminds us of metamathematics and mathematical logic. It is natural to ask logicians whether they can justify some general scheme for this kind of reasoning. And indeed, several schemes of this type are well known. Here is one of them, the so-called quantifier elimination for algebraically closed fields of characteristic 0 (Seidenberg-Tarski).

Consider formulas that contain variables (whose range is some field), addition, multiplication, equality sign, logical operations **(and, or, not, if ... then)** and quantifiers ($\forall, \exists$). Examples of formulas are

$$\exists y(xy = 1)$$

and

$$\exists x((x^2 + px + q = 0) \text{ and } (x^2 + rx + s = 0)).$$

The first formula says that element x has a multiplicative inverse; the second one says that two quadratic equations with given coefficients have a common root.

The quantifier elimination theorem guarantees that any formula is equivalent to a quantifier-free one. It is easy to see what are the quantifier-free formulas in our two examples: the first formula is equivalent to $x \neq 0$ (here inequalities come into play); the second one is true if and only if the resultant polynomial

$$det \begin{vmatrix} 1 & p & q & 0 \\ 0 & 1 & p & q \\ 1 & r & s & 0 \\ 0 & 1 & r & s \end{vmatrix}$$

is equal to zero.

By "equivalent" we mean that both formulas are simultaneously true or false for any elements of any algebraically closed field of zero characteristic. Both examples are formulas with free variables (x in the first formula; p, q, r, s in the second one). For formulas without free variables the equivalent quantifier-free formula should be a logical constant **true** or **false**, and we get the following statement ("completeness of the theory of algebraically closed fields of zero characteristic"):

Any theorem that can be expressed as a formula and proved for some algebraically dosed field of zero characteristic is automatically true for all such fields.

To show that this theorem is not something trivial, let us see why Hilbert's Nullstellensatz is a corollary. One of the forms of the Nullstellensatz says that if the system

$$\begin{cases} P_1(x_1, \ldots, x_k) = 0 \\ \ldots \\ P_n(x_1, \ldots, x_k) = 0 \end{cases}$$

has no solution $x_1, \ldots, x_k \in \mathbb{C}$, then there exist polynomials $Q_1, \ldots, Q_n$ such that $P_1Q_1 + \ldots + P_nQ_n = 1$. Let us assume for simplicity that the coefficients of P's are integers. Then the statement that the system has no solution can be expressed as a formula. Being true over $\mathbb{C}$, this formula should be true for any algebraically closed field of zero characteristic (and therefore for any field of characteristic zero, for a solution in some field remains a solution in the algebraic closure). *So it is enough to show that if Q's with the desired properties do not exist, then there is a field where the system has a solution.* But this is easy: the nonexistence of Q's means that the ideal generated by $P_1, \ldots, P_k$ is not trivial, so it is contained in some maximal ideal $I \subset \mathbb{C}[x_1, \ldots, x_k]$. Then $\mathbb{C}[x_1, \ldots, x_k]/I$ is a field where the system has solution $x_1 = [x_1], \ldots, x_k = [x_k]$,where $[x_1]$ is the image of the polynomial x_1 under the factor-mapping $\mathbb{C}[x_1, \ldots, x_k] \to \mathbb{C}[x_1, \ldots, x_k]/I$.

Exercises

1. Translate the italicized parts of the text. Pay attention to the grammatical constructions.

2. Give the Russian equivalents of the following expressions:

to be devoted to; pairwise intersection; to have to confess; to extend to; the analytic continuation principle; to point out; an absolute value; this way of reasoning; q.e.d.; to depend on; the quantifier elimination; the algebraically closed field; it is easy to see; both formulas are simultaneously true or false; let us assume; it is enough to show.

3. Translate the following:

a) **Диагональная матрица** - это квадратная матрица, все элементы которой, стоящие вне главной диагонали, равны нулю.

b) **Теорема Гамильтона-Кэли.** Любая квадратная матрица удовлетворяет своему характеристическому уравнению: если A - квадратная матрица и $c(\lambda)$ - её характеристический многочлен, то $c(A) = 0$.

4. Translate the following sentences from Russian into English using the word combinations from the text:

1) Для людей, не очень сведущих в математике, еще более странной представляется математическая логика, с помощью которой можно доказать самые невероятные утверждения.

2) Для того чтобы очаровать людей математикой, рассмотрим без подробных доказательств ряд теорем с использованием математической логики.

3) Верно ли следующее утверждение: если взять три пересекающиеся окружности, каждые две из которых имеют пересекающиеся хорды, то все хорды имеют одну общую точку?

4) Справедливо ли следующее: если какое-то геометрическое соотношение справедливо на ряде конфигураций (имеется не пустое множество), то в соответствии с принципом аналитического продолжения это соотношение справедливо всегда?

5) Одним из примеров принципа доказательства математических утверждений является теорема Гамильтона-Кэли.

6) Это равенство оказывается справедливым, потому что рассмотренные функции и операторы с различными собственными значениями образуют плотное множество.

7) Одним из основных понятий математической логики является предикат - это высказывание, в которое можно подставлять аргументы. Если аргумент один, то предикат выражает свойство аргумента, если больше – то отношение между аргументами.

8) Примером предиката является высказывания: "Сократ – человек", "Платон – человек". Оба эти высказывания выражают свойство "быть человеком". Таким образом, мы можем рассматривать предикат "быть человеком" и говорить, что он выполняется для Сократа и Платона.

9) Квантор — общее название для логических операций, ограничивающих область истинности какого-либо предиката и создающих выказывание.

5. Read the text again and write a summary.

6. Read the quotations below and answer the questions.

Hermann Walther Nernst (1864-1941), German chemist.

On examinations: Knowledge is the death of research. Nernst's motto.

Karl Pearson (1857-1936), British statistician.

All great scientists have, in a certain sense, been great artists; the man with no imagination may collect facts, but he cannot make great discoveries.

Ernest Rutherford (Baron Rutherford of Nelson) (1871-1937), New Zealand-born British physicist.

All science is either physics or stamp collecting.

1. Do you think Rutherford is prejudiced against other sciences?

2. Do you think there is any hostility between mathematics and physics?

3. Name some scientists who were successful in both physics and mathematics. What was the key to the success?

4. Do you agree with Nernst?

5. Try to explain his words.

6. Nernst was a chemist, do you think his motto can be applied to mathematics?

7. Don't you think imagination in mathematics may be useful? Where?

XIV. U-Substitution

Colin Adams, The Mathematical Intelligencer, volume 23, number 2, spring 2001, p. 29.

This is complete and utter humiliation in its nastiest form, thought the coach, as he glanced up at the scoreboard. His team was still in the single digits while their opponents were about to break fifty. He looked down the bench at his demoralized team. It was not a bright moment for the Valparaiso Variables. What had the owner James Stewart been thinking when he moved the team up to the big leagues? They weren't in any way comparable to the Indianapolis Integrals, the team that was currently scoring at will as he watched from the sideline. $\int e^{i\pi x}dx$ stole the ball from hapless z and effortlessly scored again. His players just couldn't keep up.

And not that they needed it, but the Integrals had recently signed $\int 1/(1+x^2)dx$. Here was one of the most famous integrals on the planet, with endorsement deals galore. You couldn't turn on your television without seeing $\int 1/(1+x^2)dx$ biting into a hot dog, or hawking graphing calculators. He was scoring at will. The Variables were scared of coming within 10 feet of him.

As the coach looked down his bench, all the players started down at their feet, afraid to meet his eye. They didn't want to be put into the game just to be humiliated. All except that skinny kid $\tan u$, sitting at the end. That kid's got guts, thought the coach. He'd been hustling all semester. He had two left feet, but he wanted to play so bad. And now he was looking at the coach with desperate hope in his eye. What the hell, thought the coach, this game is lost anyway.

"Okay tan u you're going in for x. You cover the big integral." Tan u leaped off the bench. The coach signaled the referee.

"I'm doing a u-subsitution," he said. "I'm replacing x with $\tan u$."

The rest of the bench looked up, startled to hear the call. x raced over to the sideline.

"What are you doing, coach? You're subbing that skinny kid in for me? Hey, I'm your top scorer. You can't do this."

"Sit down, x," said the coach, as he threw him a towel.

$\int 1/(1+x^2)dx$ laughed when he saw the scrawny player that was covering him.

"Hey look at this," he said. "They did a u-substitution." His teammates guffawed.

As play resumed, $\int 1/(1+x^2)dx$ came down the court fast, with little tan u trailing behind. But as $\int 1/(1+x^2)dx$ made a move to the right, tan u slipped to the inside. As everyone watched in amazement, the $1+x^2$ became $1+\tan^2 u$. A stunned second later, $1+\tan^2 u$ became $\sec^2 u$. The dx became $\sec^2 u du$. The crowd leapt to its feet. The $\sec^2 u$'s cancelled, and all that was left of the mighty $\int 1/(1+x^2)dx$ was $\int du$. The other integrals stood dumbfounded. The coach was waving a towel over his head. The entire Variable bench was up screaming. Do it, do it! The $\int du$ became just $u+C$. Pandemonium erupted all over the stadium.

"Okay, x, finish it off," said the coach, grinning from ear to ear. A sheepish x went back in for tan u, and the $u+C$ became $\arctan x + C$. The building reverberated with cheers. The Variables lifted scrawny $\tan u$ on their shoulders and paraded around the court as the crowd chanted, "Tan u, tan u." Fans mobbed them from all sides. The Variables had won the game.

Later in the locker room, after all the reporters had come and gone, and all the champagne had been swallowed or dumped on heads, the coach gathered the team together.

"Well, I didn't think we could do it, but thanks to $\tan u$, we beat the Indianapolis Integrals. And I want you to savor this victory. You deserve it. But don't get carried away with it, either. Next week we play the Pittsburgh PDE's, and if you think the Integrals are tough, wait until you try solving a PDE."

Exercises

1. *Specify the italicized words below and translate the sentences into Russian:*

1) *Opening* a copy of *The Mathematical Intelligencer* you *may* ask yourself uneasily, "What is this anyway – a mathematical journal or what?"

2) Relax. Breathe regularly. It's mathematical, it's a humor column, and it *may* even be harmless.

3) You couldn't turn on your television without *seeing* $\int 1/(1+x^2)dx$ *biting* into a hot dog, or *hawking* graphing calculators.

4) This is complete and utter humiliation in its nastiest form, thought the coach, *as* he glanced up at the scoreboard. His team was still in the single digits *while* their opponents were about to break fifty.

5) They weren't in any way comparable to Indianopolis Integrals, the team that was currently scoring at will *as* watched from the sideline.

2. *What meaning do the italicized verbal forms convey?*

1) And not that they needed it, but the Integrals *had* recently *signed* $\int 1/(1+x^2)dx$.

2) That kid's got guts, thought the coach. He'*d been hustling* all semester.

3) $\int 1/(1+x^2)dx$ laughed when he saw the scrawny player that *was covering* him.

4) Later in the locker room, after all the reporters *had come* and *gone*, and all the champagne *had been swallowed* or *dumped* on heads, the coach gathered the teams together.

5) What *had* the owner James Stewart *been thinking* when he moved the team up to the big leagues?

3. *Find the following phrasal verbs in the text and try to guess their meaning.*

to be about
to go in for
to look up
to move up
to turn on

Now use them to complete the sentences below:

1. He ... ... to take the floor when the phone rang. 2. The lecturer ... ... from his notes as silence fell on the audience. 3. Some people ... ... team sports, while others prefer individual sports. 4. Could you ... ... a bit? 5. I'll ... ... the heating.

4. *Give the English equivalents of the following Russian terms. Make sure that you know how to spell them and how they are abbreviated.*

Переменная, интеграл, трансцендентное число, производная, фут, тангенс, арктангенс, котангенс, секущая.

5. *Using your English-English dictionary, say what words the following derive from and specify their suffixes and prefixes. Mind the pronunciation.*

Uneasily, mathematical, comparable, hapless, effortlessly, referee, replace, mighty, sheepish, amazement, harmless, disorientation, calculator, reverberate.

6. *Add the missing letters to complete the words below. Use your dictionary to check your answers, if necessary.*

U...ter, jo...rnal, g...lore, desper...te, uneas...ly, substitu...ion, fi...ty, leag...e, integr...l

7. *For each group of sentences think of one word only which can be used appropriately in all the three sentences.*

1) There were 100 students in the lecture hall, at the very ... .
... rock music makes me feel irritated.

It is a ... interesting exhibition. You must certainly go and see it.

2) He has an ... sense of hearing.

He was taken to hospital with ... appendicitis.

An ... angle is an angle that is less than 90 degrees.

3) It's always better to look on the ... side of things.

He is always full of ... ideas.

It was a ... moment for the young musician.

4) In any ..., you are right.

First, we visited France and then we went to Italy. – No, it was the other ... round.

Preparations are under ... for our conference in April.

5) Where there is a ... there is a way.

Candidates are not allowed to go out at ... during the exam.

With the best ... in the world he couldn't become a linguist simply because he had no knack for learning languages.

6) In the ... they found their way through the bushes.

There was an interesting episode at the ... of the book.

The students listened to the lecture until the very ....

8. Synonyms are words with *roughly* the same meaning. We say roughly because it is hardly possible to find any two synonyms that have *exactly* the same meaning: either there is a shade which separates them or not all synonyms collocate with some word.

In the text you have come across several synonyms of the verb 'surprise'. Below there is a note on how to use the verb and its synonyms.

The note is borrowed from the website of the

Oxford Advanced Learners Dictionary

Startle, amaze, stun, astonish, take somebody aback, astound

These words all mean to make somebody feel surprised.

surprise to give somebody the feeling that you get when something happens that you do not expect or do not understand, or something that you do expect does not happen; to make somebody feel surprised: *The outcome didn't surprise me at all.*

startle to surprise somebody suddenly in a way that slightly shocks or frightens them: *Sorry, I didn't mean to startle you. The explosion startled the horse.*

amaze to surprise somebody very much: *Just the huge size of the place amazed her.*

stun (rather informal) (often in newspapers) to surprise or shock somebody so much that they cannot think clearly or speak astonish to surprise somebody very much: *The news astonished everyone.*

amaze or astonish?

These two words have the same meaning and in most cases you can use either. If you are talking about something that both surprises you and makes you feel ashamed, use ***astonish***: *He was astonished by his own stupidity.*

take somebody aback [usually passive] (especially of something negative) to surprise or shock somebody: *We were rather taken aback by her hostile reaction.*

astound to surprise or shock somebody very much: *His arrogance astounded her.*

It surprises somebody/startles somebody/amazes somebody/stuns somebody/astonishes somebody/takes somebody aback/astounds somebody

to surprise/startle/amaze/stun/astonish/astound somebody ***that***...

to surprise/amaze somebody *what/how*...

to surprise/startle/amaze/stun/astonish/astound somebody to know/find/learn/see/hear...

to be surprised/startled/stunned into (doing) something

9. *Read the quotations below. Answer the questions.*

Max Born (1882-1970), German physicist.

To present a scientific subject in an attractive and stimulating manner is an artistic task, similar to that of a novelist or even a dramatic writer. The same holds for writing textbooks.

Francis Henry Compton Crick (1916-2004), British molecular biologist.

There is no form of prose more difficult to understand and more tedious to read than the average scientific paper.

1. What makes a good textbook?

2. What is the difference between a good book and a bad book? Give examples.

3. Could you name some really good (bad) textbooks on mathematics or mechanics? Do the other students agree with you?

John Henry Newmann (1801-90), British cardinal and theologian.

To discover and to teach are distinct functions; they are also distinct gifts, and are not commonly found united in the same person.

4. What made Newmann think so?

5. Why is it possible?

6. Do you think a person is unlikely to have both gifts?

XV. The Surfaces of Delaunay

James Eells, Mathematical Intelligencer, Volume 9, No. 1 (Winter 1987), pp. 53-57.

I. Background

In 1841 the astronomer/mathematician C. Delaunay isolated a certain class of surfaces in Euclidean space, representations of which he described explicitly. In an appendix to that paper, M. Sturm characterized Delaunay's surfaces variationally; indeed, as the solutions to an isoperimetric problem in the calculus of variations. That in turn revealed how those surfaces make their appearance in gas dynamics; soap bubbles and stems of plants provide simple examples. See Chapter V of the marvellous book [8] by D'Arcy Thompson for an essay on the occurence and properties of such surfaces in nature.

More than 130 years later E. Calabi pointed out to me that the solutions to a certain pendulum problem of R. T. Smith could be interpreted via the Gauss maps of Delaunay's surfaces. And Eells and Lemaire found that the Gauss map of one of those surfaces produces a solution to an existence problem in algebraic/differential topology.

The purpose of this article is to retrace those steps in an expository manner - as a revised version of [2].

II. Roulettes of a Conic

The first step is to derive the equations describing the trace of a focus F of a non-degenerate conic l as K rolls along a straight line in a plane. (Perhaps these derivations were better known a century ago!) We examine various cases separately.

L IS A PARABOLA (SEE FIGURE 13-1) Here A is the vertex of l. The line PK is tangent to l at the point K. The following properties are elementary: 1. Correspondingly marked angles are equal. 2. FP is orthogonal to PK.

Thus we obtain

$$FA = FP \cos \angle AFP = FP \cos \angle PFK$$

Now we change our viewpoint and think of the tangent line PK as the axis – the x-axis – along which the parabola l rolls. We denote the ordinate of F by y and observe that

$$\cos \angle PFK = \frac{dx}{ds}$$

describes the rate of change of abscissa of F with respect to arc length s; i.e.,

$$\frac{dx}{ds} = \alpha,$$

where α denotes the angle made by the tangent with the x-axis. Thus setting $c = FA$, we obtain the differential equation

$$c = y\frac{dx}{ds} = \frac{y}{\sqrt{1+y^2}}, \text{ or } y = \sqrt{\frac{y^2 - c^2}{c^2}}.$$

Its solution is the *catenary*

$$y = \frac{c}{2}(e^{x/c} + e^{-x/c}) = c \cosh x/c$$

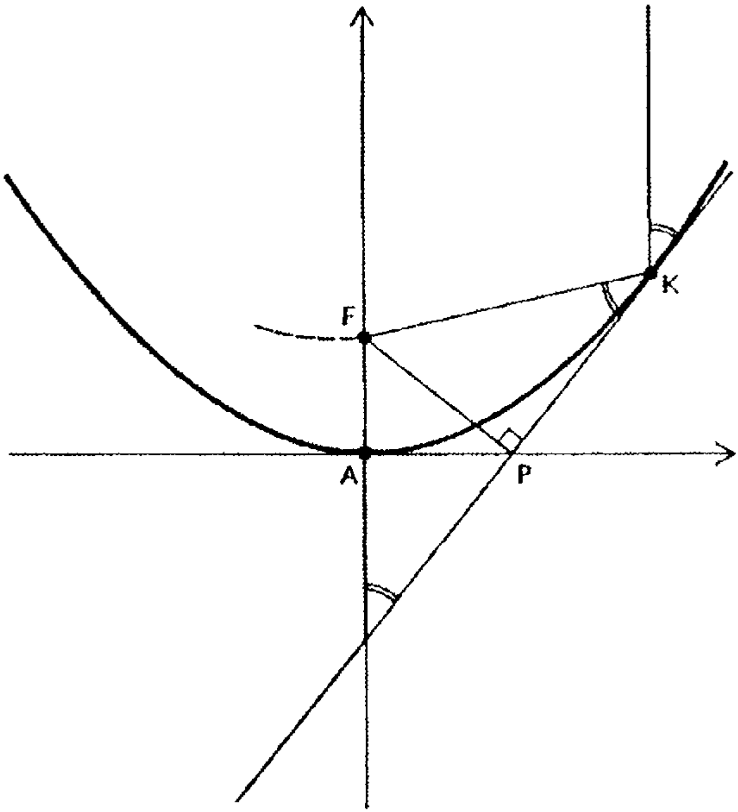

FIGURE 13-1 ℓ *is a parabola*

That equation describes the shape of a flexible inextensible free-hanging cable – thereby explaining its name. In that context we can obtain the equation of the catenary as the Euler-Lagrange equation of the potential energy integral

$$P(y) = \int_{x_0}^{x_1} y\sqrt{1+y'^2}dx$$

subject to vatiations holding fixed the length integral

$$\int_{x_0}^{x_1} y\sqrt{1+y'^2}dx = L$$

Indeed, from general principles we are asked to find a real number a and an extremal of the integral

$$J(y) = \int_{x_0}^{x_1} (\sqrt{1+y'^2} + ay\sqrt{1+y'^2})dx$$

Its Euler-Lagrange equation has first integral

$$y' = \sqrt{\frac{(1+ay)^2 - b^2}{b^2}} \text{ for } b \in \mathbb{R}$$

The equation of the catenary is derived from this, choosing suitable normalizations. The curvature of l is measured by the amount of turning of its tangent. That is expressed by the *Gauss map* of l into the unit circle, given by $x \to \alpha_x$, where

$$\cos\alpha_x = \frac{dx}{ds} = \frac{c}{y}.$$

The Gauss map of the roulette of the parabola is injective onto an open semi-circle.

l IS AN ELLIPSE Here F and F' are the foci of l; the O is its center. The line PKP' is tangent to l at K. Letting a and b denote the length of the semi-axes of l, we obtain the following properties:

1. $FK + F'K = 2a > 0$;
2. the pedal equation $PF * P'F' = b^2$;

3. the normal to the locus of F passes through K.

Again using PK as x-axis,

$$\frac{y}{FK} = \sin\angle FKP = \cos\angle FTP = \frac{dx}{ds}$$

$$\frac{y'}{F'K} = \sin\angle F'KP' = \cos\angle F'TP' = \frac{dx}{ds}.$$

From these we derive

$$y + y' = 2a\frac{dx}{ds}, yy' = b^2,$$

so that

$$y^2 - 2ay\frac{dx}{ds} + b^2 = 0.$$

By analyzing all cases and taking $a \leq b$, we obtain

$$y^2 \pm 2ay\frac{dx}{ds} + b^2 = 0.$$

The solutions to that differential equation can be given explicitly in terms of elliptic functions. The locus of either focus will be called the *undulary*. Its Gauss map is given by $x \to \alpha_x$, where

$$\cos\alpha_x = \mp\frac{y^2 + b^2}{2ay}.$$

It maps l onto a closed arc of the unit circle. There are two limiting cases, which are perhaps best handled separately. When $b \to a$ the undulary degenerates to a straight line, the locus of the centre of a circle rolling on a line. And where $b \to 0$ the undulary becomes a semi-circle centered on the x-axis.

l IS AN HYPERBOLA In analogy with the case of the ellipse, we have 1. $FK - F'K' = 2a > 0$. 2. $PF - P'F' = b^2$. Thus we obtain the following differential equation for the locus of F, given as a first integral of an Euler-Lagrange equation:

$$y^2 \pm 2ay\frac{dx}{ds} - b^2 = 0.$$

The loci of the two foci fit together to form the curve which we shall call the *nodary*. Its Gauss map $x \to \alpha_x$ is governed by

$$\cos \alpha_x = \mp \frac{y^2 - b^2}{2ay}.$$

The Gauss map has no extreme points, and direct verification shows that it is surjective. A *roulette of a conic* is a catenary, undulary, nodary, a straight line parallel to the x-axis, or a semicircle centered on the x-axis.

III. Surfaces of Revolution with Constant Mean Curvature

Rotating each of the roulettes about its axis of rolling produces five types of surfaces in Euclidean 3-space $\mathbb{R}^3$, called *the surfaces of Delaunay:* the *catenoids, unduloids, nodoids,* the *right circular cylinders*, and the *spheres.*

VARIATIONAL CHARACTERIZATION: *We formulate the following isoperimetric principle, for the unduloid and nodoid (only minor technical changes being required for the other cases).* Consider graphs in $\mathbb{R}^2$ of non-negative functions

$$y : [x_0, x_1] \to \mathbb{R}(\geq 0)$$

with fixed volume of revolution

$$V(y) = \pi \int_{x_0}^{x_1} y^2 dx;$$

and extremize their lateral area

$$A(y) = 2\pi \int_{x_0}^{x_1} y^2 ds$$

holding the endpoints fixed. By general principles of constraint (under the heading of Lagrange's method of multipliers for isoperimetric problems), we are led to the Euler-Lagrange equation associated with the integral

$$F(y) = \pi \int_{x_0}^{x_1} (y^2 dx + 2ay\, ds) = \pi \int_{x_0}^{x_1} (y^2 + 2ay\sqrt{1+y'^2}dx).$$

Here a is a convenient real parameter. Its integrand f does not involve x explicitly, so we obtain a first integral from

$$0 = y'(f_y - \frac{d}{dx}f_{y'}) = \frac{d}{dx}(f - y'f_{y'}).$$

Thus $f - y'f_{y'} = \pm b^2$, where b is another real parameter. Consequently,

$$y^2 + \frac{2ay}{\sqrt{1+y'^2}} \mp b^2 = 0.$$

But

$$\frac{1}{\sqrt{1+y'^2}} = \frac{dx}{ds}$$

so the extremal equation for our variational problem coincides with that of the roulette of the ellipse or hyperbola.

GAUSS MAPS: In an analogy with the case of oriented curves in the plane, we associate to any oriented surface M immersed in $\mathbb{R}^3$ its Gauss map $\gamma : M \to S$ (the unit 2-sphere centered at the origin in $\mathbb{R}^3$), defined by assigning to each point $x \in M$ the positive unit vector orthogonal to the oriented tangent plane to M at x. Its differential $d\gamma(x)$ can be interpreted as a symmetric bilinear form on the tangent space T_xM. Its eigenvalues λ_1, λ_2 are well determined up to order. The symmetric functions $K_x = \lambda_1\lambda_2$ and $H_x = (\lambda_1\lambda_2)/2$ are called the *curvature* of M and the *mean curvature of the immersion* at x, respectively. For instance,

1. the cylinder has $K \equiv 0$ and constant mean curvature $H \neq 0$;

2. the sphere of radius R has a constant curvature $K = 1/R^2$ and constant mean curvature $H = 1/R$;

3. the catenoid has variable curvature K and mean curvature $H \equiv 0$;

4-5. the unduloid and nodoid have variable curvature K and constant mean curvature $H \neq 0$.

These five surfaces were recognized by Plateau, using soap film experiments.

Say that a surface of constant mean curvature in $\mathbb{R}^3$ is *complete* if it is not part of a larger such surface. From Sturm's variational characterization, we obtain

DELAUNAY'S THEOREM: *The complete immersed surfaces of revolution in $\mathbb{R}^3$ with constant mean curvature are precisely those obtained by rotating about their axes the roulettes of the conics.*

Thus Delaunay's surfaces are those surfaces of revolution M in $\mathbb{R}^3$ which are maintained in equilibrium by the pressure of a field of force which acts everywhere orthogonally to M.

IV. Harmonic Gauss Maps

An easy yet vitally important theorem of Ruh-Vilms states that:

A surface M immersed in R^3 has constant mean curvature if and only if its Gauss map $\gamma : M \to S$ satisfies the equation

$$\Delta\gamma = ||d\gamma||^2\gamma,$$

where Δ denotes the Laplacian of M with conformal structure induced from that of $\mathbb{R}^3$, and vertical bars the Euclidean norm at each point. Indeed, (1.4) *is the condition for harmonicity of the map γ* — and is the Euler-Lagrange equation associated to the *energy (or action) integral*

$$E(\gamma) = \frac{1}{2}\int_M ||d\gamma||^2.$$

E is a conformal invariant of M.

SMITH'S MECHANICS: *Motivated by certain mechanical analogies, R. T. Smith found solutions to equation (1.4) as maps: $\gamma : \mathbb{R}^2 \to S$, as follows:*

Think of points of $\mathbb{R}^2$ parametrized by angles (ϕ, θ), and use spherical coordinates on the sphere S, as shown in Figure. If we restrict our attention to maps γ of the special form

$$(\phi, \theta) = (e^{i\theta}\sin\alpha(\phi), \cos\alpha(\phi)),$$

then the equation of harmonicity becomes the pendulum equation

$$\alpha" = \frac{A}{2} \sin 2\alpha. \quad (4.3)$$

We assume that $\alpha(0) = \pi/2$, so that the solution oscillates symmetrically about $\pi/2$. Now a first integral of (4.3) is given by

$$\alpha' = \sqrt{\frac{C - A\cos^2\alpha}{2}}.$$

Again, that has an explicit solution in terms of elliptic functions. *Furthermore, the associated map $\gamma : \mathbb{R}^2 \to S$ is doubly periodic, factoring through the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ to produce a map $\gamma : T \to S$, as desired.* Incidentally, the integrand of E is

$$\| d\gamma \|^2 = \alpha'^2 + \frac{A}{2} \sin^2 \alpha.$$

Calabi made the beautiful observation that *Smith's maps $\gamma : T \to S$ are the Gauss maps of certain surfaces of Delaunay.*

A HARMONIC REPRESENTATIVE IN A HOMOTOPY CLASS: If we represent the torus T in the form $T = \mathbb{R}/a\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ and use polar coordinates (r, θ) on the unit sphere S, then a map from the cylinder to S of the form

$$r = \Phi(x), \theta = y$$

subject to the coordinates $\Phi(0) = 0, \Phi(a) = \pi$ is harmonic if and only if Φ satisfies the pendulum equation (4.3) with $A = 1$. There are such solutions. Indeed, *the Gauss map of the nodoid induces a harmonic map of a Klein bottle $\gamma : K \to S$. Furthermore, that map is not deformable to a constant map.*

Hopf's classification theorem insures that the maps $K \to S$ are partitioned by homotopy into just two classes. Thus the harmonic map γ represents the non-trivial class.

Exercises

1. *In the text, translate the sentences italicized.*

2. *What key words and phrases would you use to speak about roulettes of a conic?*

3. *See if you remember:* in turn, with respect to, via, in terms of, in analogy with, consequently, correspodingly, i.e., respectively, incidentally.

4. *Complete the expressions with a word which can combine with the words given:*
to obtain ... , to derive ... , to satisfy ... , the Euler-Lagrange ..., the extremal ... coincides with solutions to ... .

5. *Insert prepositions:*
a solution ... a problem, to be tangent ... l ... the point k, to think ... smth, subject ... smth, to map smth ... smth, to degenerate ... a straight line, to rotate smth ... its axis, a line parallel ... the x-axis, ... the heading of Lagrange's method ... multipliers .. isoperimetric problems, to be led ... smth, the positive unit vector orthogonal ... smth, as shown ... Figure 1, to restrict attention ... smth, to be partitioned ... classes.

6. *What are the plural forms of the following nouns? Use a dictionary to help.*
Focus, axis, pendulum, locus, abscissa.

7. *Use your dictionary to check the pronunciation of* **eigenvalue***.*

8. *What do you understand by the term* **nodary**?

9. *Give the English equivalents of:*
1) мы полагаем, что
2) если мы представим тор T в форме
3) тогда и только тогда

4) симметрично относительно $\pi/2$
5) скажем, что
6) мы получим
7) которые остаются в равновесии
8) сила, которая действует повсеместно
9) простая, но чрезвычайно важная теорема гласит, что
10) если отображение удовлетворяет уравнению
11) уравнение, имеющее отношение к
12) как следует из
13) как показано на рисунке
14) если мы сосредоточим наше внимание на
15) как требуется
16) уравнение маятника
17) следовательно
18) таким образом мы получим
19) непосредственная проверка показывает, что
20) прямая параллельная оси x

10. *Read the quotations below and answer the questions.*

Irving Langmuir (1881-1957), American chemist and physicist.

A chemist who does not know mathematics is seriously handicapped.

Arnold Sommerfeld (1868-1951), German physicist.

If you want to be a physicist, you must do three things - first, study mathematics, second, study more mathematics, and third, do the same.

Alfred North Whitehead (1861-1947), British mathematician and philosopher.

All science as it grows toward perfection becomes mathematical in its ideas.

1. Why does mathematics become increasingly important in many fields nowadays?

2. Mathematics is employed by biology, psychology, linguistics, etc. Could you give any examples of its applications?

Napoleon Bonaparte (1769-1821), French emperor.

The advancement and perfection of mathematics are intimately connected with the prosperity of the State.

3. What do you think made Napoleon say so?

4. Try to give examples supporting the quotation.

George-Louis Leclerc, Comte de Buffon (1707-88), French naturalist and philosopher.

To be and to think are one and the same for us.

1. Do you agree with this statement?

2. Is this quotation similar to René Descartes' "Cogito ergo sum"?

XVI. π is wrong!

Bob Palais, The Mathematical Intelligencer, volume 23, number 3, summer 2001, pp. 7-8.

I know it will be called blasphemy by some, but I believe that π is wrong. For centuries π has recieved unlimited praise; mathematicians have waxed rhapsodic about its mysteries, used it as a symbol for mathematics societies and mathematics in general, and built it into calculators and programming languages. Even a movie has been named after it. *(For a non-technical movie, the mathematics was surprisingly good, except for the throwaway question "Surely you've tried all of the 216-digit numbers?" At one number per nanosecond, checking all 30-digit numbers would take longer than the life of the universe!)* I am not questioning its irrationality, transcendence, or numerical calculation, but the choice of the number on which we bestow a symbol conveying deep geometric significance. *The proper value, which does deserve all of the reverence and adulation bestowed upon the current impostor, is the number now unfortunately known as* 2π.

I do not necessarily feel that π can or even should be changed or replaced with an alternative (though I've by now recieved some good suggestions!), but it is worthwhile to recognize the repercussions of the error as a warning and a lesson in choosing good notational conventions to communicate mathematical ideas. *I compare the problem to what would have occurred if Leonard Euler had defined e to be* $.3678\ldots$ *(the natural decay factor equal* *to* $\frac{1}{2.718...}$*), in which case there would be just as many unfortunate minus signs running around from that choice as there are factors of 2 from* $\pi = 3.14\ldots$.

The most significant consequence of the misdefinition of π is for early geometry and trigonometry students who are told by mathematicians that radian measure is more natural than degree measure. *In a sense it is, since a quarter of a circle is more naturally measured by* $1.57\ldots$ *than by 90.* Unfortunately, this beautiful idea is sabotaged by the fact that π isn't $6.28\ldots$, which would make a quarter of a circle or a *quad*rant equal to a quarter of π radians; a third of a circle, a third of π radians, and so on. The opportunity to impress students with a beautiful and natural simplification is turned into an absurd exercise in memorization and dogma. *An enlightening analogy is to leave clocks the way they are but define an hour to be 30 minutes. In that case, 15 minutes or a quarter of a clock would indeed be called half an hour, just as a quarter of a circle is half of* π *in mathematics!* Even mathematically sophisticated software packages prefer to use 90^o to indicate a quarter-circle rotation. We can't really blame them for the fact that π is wrong.

Perhaps more convincing to mathematicians is the litany of important theorems and formulas into which this ubiquitous factor of 2 has crept and propagated: Cauchy's integral formula and Fourier series formulas all begin with $\frac{1}{2\pi}$, Stirling's approximation and the Gaussian normal distribution both carry it, the Gauss-Bonnet and Picard theorems have the mark of 2π. (Archimedes showed that the area of the unit sphere is the area of the cylinder of the same radius and height, or twice the circumference of the unit circle: $4\pi = 2(2\pi)$.) The blight of factors of 2 even affects physics, for example in Maxwell's equations (Gauss's law, Ampere's law, Coulomb's constant) and Planck's constant $\frac{h}{2\pi}$. Euler's formula *should* be $e^{i\pi} = 1$ (or $e^{i\pi/2} = -1$, in which case it involves one more fundamental constant, 2, than before). Wouldn't it be nicer if the periods of the fundamental circular functions cos and sin were π rather than 2π? *Wouldn't it be nicer if half-plane integrals such as the Hilbert transform were indicated by the appearance of a factor of 2 rather than its disappearance?*

The sum of the interior angles of a triangle is π, granted. But the sum of the *exterior* angles of *any* polygon, from which the sum of the interior angles can easily be derived, and which generalizes to the integral of the curvature of a simple closed curve, is 2π. The natural formula for area of a circle, $\frac{1}{2}\pi r^2$, has the familiar ring of $\frac{1}{2}gt^2$ or $\frac{1}{2}mv^2$; it would have instilled good habits for representing quadratic quantities and foreshadowed the connection between the area of a circle and the integral of circumference (with respect to *radius*) better than πr^2. Another way of putting it is that radius is far more convenient than diameter – consider what the unit circle means. *If it weren't, I would agree that the traditional choice of* π *was right.*

Of course you may say that none of this really matters or affects the mathematics, because we may define things however we like; and that is correct. But the analogy with e mentioned above, or the idea of redefining the symbol i to mean $\frac{\sqrt{-1}}{2}$ shows the true folly of π. *Neither of these changes would change the mathematics, but nor would anyone deny they are absurd.*

What really worries me is that the first thing we broadcast to the cosmos to demonstrate our "intelligence," is $3.14\ldots$. *I am a bit concerned about what the lifeforms who recieve it will do after they stop laughing at creatures who must rarely question orthodoxy.* Since it is transmitted in binary, we can hope that they overlook what becomes merely a bit shift!

I would not be surprised and would be interested to hear if this idea has been discussed previously, but I was unable to find any reference either in the wonderful *Pi: A Source Book* by Lennart Berggren, Jonathan Borwein, and Peter Borwein, or in Petr Beckmann's *A History of Pi*, or on the Internet. When I have suggested to people that π has a flaw, their reactions range from surprise, amusement, and agreement, to "Of course, I knew it all along," to dismissal, to indignation.

The history (I was surprised, along with everyone I tell, that the symbol was not in use in ancient Greece): Oughtred used the symbol π/δ in 1647 for the ratio of the periphery of a circle to its diameter. David Gregory (1697) used π/ρ for the ratio of the periphery of a circle to its radius. The first to use π as we use it now was a Welsh mathematician, William Jones, in 1706 when he stated 3.14159 & c. $=\pi$. Euler, who had until then been using the letters p and c, adopted the symbol in 1737, leading to its universal acceptance. *If only he or Jones had set Gregory's ρ to be 1 instead of Oughtred's δ, our formulas today would be much more elegant and clear.*

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$1\ \pi\!\!\pi = 2\pi = 6.283\ldots$ is called **One turn**.

So instead of 90^o, the angle of a quadrant and a quarter an hour being $\frac{\pi}{2}$ ('Pi over two'), it becomes $\frac{1}{4}\ \pi\!\!\pi$, or quite naturally, 'A quarter turn'!

Many other formulas simplify:

$\cos(x+\pi\!\!\pi) = \cos(x)$ $\sin(x+\pi\!\!\pi) = \sin(x)$ — Cos, Sin Periods

$A = \frac{1}{2}\pi\!\!\pi r^2$ — Area, ($\frac{1}{2}mv^2$, $\frac{1}{2}gt^2$),

etc, like Striling's Formula, Euler's Formula, Dirac's Constant, Angular Frequency, Fourier Coefficients, Cauchy's Formula, Gaussian Distribution, Nth Roots Of Unity.

Exercises

1. *In the text, translate the sentences italicized.*

2. *Insert prepositions:*

mathematics ... general, to name smth (smb) ... smth (smb), one number ... nanosecond, to replace smth ... an alternative, to compare smth ... smth, to impress smb ... smth, to blame smb ... smth, to generalize smth ... smth ... , integral of cirumference .... respect ... radius, ... the Internet, ratio of the periphery of a circle ... its diameter, contribution ... this paper.

3. *Give the English equivalents of:*

1) математика в целом
2) стоит признать
3) множитель двойки

4) важное следствие

5) в некотором смысле

6) убедительный

7) нормальное распределение

8) площадь единичной сферы

9) внутренний угол

10) варьироваться от ... до ...

11) корень n-ой степени из единицы

4. *Using your English-English dictionary, check the pronunciation and stress of the words below:*

society, significance, sign, consequence, quadrant, ubiquitous, series, area, sphere, height, circumference, interior, exterior, polygon, curvature, cosmos, ratio, neither.

5. *Make sure you pronounce the following surnames correctly:*

Euler, Cauchy, Stirling, Bonnet, Picard, Archimedes, Maxwell, Gauss, Ampere, Coulomb, Hilbert, Oughtred, Gregory, Dirac, Fourier

6. *Give the correct pronunciation of the following:*

$\pi, e, \cos, \sin, i, \delta, \rho$.

7. *At the end of the paper the author names formulas that simplify. Write them explicitly (with* ϖ*) and read them.*

8. *Answer the questions:*

1) Do you agree that forumlas look better with ϖ?

2) Could you give any formulas that would become "worse" if ϖ is introduced?

3) Could you explain the minus signs if e is replaced by $\frac{1}{e} = 0.3678\ldots$? Write down some formulas.

4) Do you think there are other unfortunate notational conventions? Give examples. Try to improve them.

9. *Read the quotations below and answer the questions.*

Thomas John Watson, Sr. (1874-1956), CEO of IBM.

I think that there is a world market for about five computers.

Dorothy M. Wrinch (1894-1976), British mathematician, biologist and chemist.

First they said my [cycol] structure [of proteins] couldn't exist. Then when it was found in Nature they said it couldn't be synthesized in a laboratory. Then when it was synthesized they said it wasn't important anyway.

1. Can you explain why Watson was so wrong?

2. Was it possible for one to predict the future of computers?

3. Could you make your prediction of IT development?

4. Explain why Wrinch's colleagues were wrong.

5. What made them think in the way they did?

6. Give other examples of false predictions.

XVII. Mathematical Anecdotes

Steven G. Krantz, The Mathematical Intelligencer, Volume 12, No. 4 (Fall 1990), 32-38.

In any field of human endeavor, the "great" participants are distinguished from everyone else by the arcana and apocrypha that surround them. Stories about Wolfgang Amadeus Mozart abound, yet there are few stories about his musical contemporaries. Mozart had the *je ne sais quoi* that made people want to tell stories about him.

And so it goes with mathematicians. Over the years I have collected dozens of anecdotes about famous mathematicians (a necessary condition for being the subject of legend is fame; it is by no means sufficient). These stories are of several types: (i) incidents to which I have been witness (there are few of these); (ii) incidents related to me by someone who witnessed them (on statistical grounds, one expects a greater number of these); and (iii) incidents that have been passed down through iterated tellings and are therefore unverifiable. I shall not consistently classify the stories that I will relate here. In many cases I cannot remember which of the three types they are, and actually knowing would generally spoil the fun. In any event, I must bear the ultimate responsibility for the stories.

In writing this article, I am running the risk that readers will think me flip, disrespectful, or (worse) that I am attacking people who cannot fight back. Let me set the record straight once and for all: To me, the mathematicians described here are among the gods of twentieth-century mathematics. Much of what we know, and certainly much of my own work, follows from their insights. *The enormous scholarly reputations of these men sometimes cause their humanity to be forgotten.* Bergman, Besicovitch, Gödel, Lefschetz, and Wiener were not merely collections of theorems masquerading as people; they had feet of clay like the rest of us. In telling stories about them, we bring them back to life and celebrate their careers.

Bergman

Stefan Bergman (1898 — 1977) was a native of Poland, He began his career in the United States at Brown University. It is said that shortly after Bergman and his mistress arrived in the United States, he took her aside and told her, "Now we are in the United States where customs are different. When we are with other people, you should call me 'Stefan.' But at home you should continue to call me 'Professor Doktor Bergman.' " Others who knew Bergman will say that he was not the sort of man who would have had a mistress. It is more likely that the man in question was von Mises (Bergman's sponsor); there is general agreement that the woman was Hilda Geiringer. In fact another story holds that Norbert Wiener (more on him later) went to D. C. Spencer around this time and said, "I think that we should call the FBI (Federal Bureau of Investigation)." Puzzled, Spencer asked why. "Because von Mises has a mistress," was the serious reply.

After a few years Bergman moved to Harvard and then to Stanford, where he spent most of his career. Supported almost always on grants and other soft money, Bergman rarely taught. This fact may have contributed to the general murkiness of his verbal and written communications. *Murkiness aside, Bergman was proud of his ability to express himself in many tongues. Said he, "I speak twelve languages – English the bestest."*

In fact Bergman had a stammer and was sometimes difficult to understand in any language. Once he was talking to Antoni

Zygmund, another celebrated Polish analyst, in their native tongue. After a bit Zygmund said, "Please let's speak English. It's more comfortable for me."

Although Bergman had many fine theorems to his credit (including the invention of a version of the Silov boundary), the crowning achievement of his mathematical work was the invention of the kernel function, now know as the Bergman kernel. He spent most of his life developing properties and applications of the Bergman and the associated Bergman metric. *It must have been a special source of pride and pleasure for him when, near the end of Bergman's life, Charles Fefferman (1974) found a profound application of the Bergman theory to the study of biholomorphic mappings.* Fefferman's discoveries, coupled with related ideas of J.J.Kohn and Norberto Kerzman, created a renaissance in the study of the Bergman kernel. Indeed a major conference in several complex variables was held in Williamstown in 1975 in which many of the principal lectures mentioned or discussed the Bergman kernel.

Bergman had always felt that the value of his ideas was not sufficiently appreciated. He attended the conference and commented to several people how pleased he was that his wife (also present) could see his work finally being recognized. I sat next to him at most of the principal lectures. In each of these, he listened carefully for the phrase "and in 1922 Stefan Bergman invented the kernel function." Bergman would then dutifully record this fact in his notes — and nothing more. *I must have seen him do this twenty times during the three-week conference.*

There was a rather poignant moment at the conference. In the middle of one of the many lectures on biholomorphic mappings, Bergman stood up and said, "I think you people should be looking at representative coordinates (also one of Bergman's inventions)". Most of us did not know what he was talking about, and we ignored him. He repeated the comment a few more times, with the same reaction. Five years later S. Webster, S. Bell, and E. Ligocka found astonishing simplifications and extensions of the known results about holomorphic mappings using — guess what? — representative coordinates.

Bergman was an extraordinarily kind and gentle man. He went out of his way to help many young people begin their careers, and he made great efforts on behalf of Polish Jews during the Nazi terror. He is remembered fondly by all who knew him. But he was a shark when it came to his mathematics. When he attended a lecture about a theorem he liked, he often went to the lecturer afterwards and said "I really like your theorem. It reminds me of my studies of the kernel function. Consider complex two-space..." And Bergman was off and running on his favorite topic. On another occasion a young mathematician gave Bergman a manuscript he had just written. Bergman read it and said "I like your result. Let's make it a joint paper, and I'll write the next one."

Whenever someone proved a new theorem about the Bergman kernel or Bergman metric, Bergman made a point of inviting the mathematician to his house for supper. Bergman and his wife were a gracious host and hostess and made their guest feel welcome. However, after supper the guest had to pay the piper by giving an impromptu lecture about the importance of the Bergman kernel.

Bergman's wife Edy was very devoted to him, but life with Stefan was sometimes trying. When they first got married, Bergman had just completed a difficult job search. In the days immediately following World War II, jobs were scarce, and Bergman wanted a position with no teaching. After a long period of disappointment, Shiffer got Bergman a position at Stanford; so the mood was high at the Bergmans' wedding reception. The reception took place in New York City, and Bergman was delighted that one of the guests was a mathematician from New York University with whom he had many mutual interests. They got involved in a passionate mathematical discussion and after a while Bergman announced to the guests that he would be back in a few hours: He had to go to NYU to discuss mathematics. On hearing this, Shiffer turned to Bergman and said "I got you your job at Stanford; if you leave this reception, I will take it away." Bergman stayed.

Bergman thought intensely about mathematics and cared passionately about his work. One day, during the 1950 International Congress of Mathematicians in Cambridge, Bergman had a luncheon date with two Italian friends. Right on schedule they appeared at Bergman's office: the distinguished elder Italian mathematician Piccone (bearing a bouquet of flowers for Bergman!) and his younger colleague Sichera. This was Piccone's first visit to the United States, and he spoke no English; Sichera acted as interpreter. After greetings were exchanged, Bergman asked Sichera whether he had read Bergman's latest paper. Sichera allowed that he had, and that he thought it was very interesting. However, he said that he felt that certain additional differentiability assumptions were required. Bergman said, "No, no, you don't understand," and proceeded to explain on the blackboard, Piccone, understanding none of this, waited patiently. After the explanation, Bergman asked Sichera whether he now understood. Sichera said that he did, but he still thought that some differentiability hypotheses were required in a certain step. Bergman became adamant and a heated argument ensued — Piccone comprehending none of it. After some time, Sichera said, "Well, let's forget it and go to lunch." Bergman cried, "No differentiability — no lunch!" and he remained in his office while the two Italians went to lunch. Piccone gave the flowers to the waitress.

There is considerable evidence that Bergman thought about mathematics constantly. Once he phoned a student, at the student's home number, at 2:00 A.M. and said, "Are you in the library? I want you to look something up for me!"

On another occasion, when Bergman was at Brown, one of Bergman's graduate students got married. The student planned to attend a conference on the West Coast, so he and his new bride decided to take a bus to California as a sort of makeshift honeymoon. There was a method in their madness: the student knew that Bergman would attend the conference but that he liked to get where he was going in a hurry. The bus seemed the least likely mode of transportation for Bergman. But when Bergman heard about the impending bus trip, he thought it a charming idea and purchased a bus ticket for himself. The student protested that this trip was to be part of his honeymoon, and that he could not talk mathematics on the bus. Bergman promised to behave. When the bus took off, Bergman was at the back of the bus and, just to be safe, Bergman's student took a window seat near the front with his wife in the adjacent aisle seat. But after about ten minutes Bergman got a great idea, wandered up the aisle, leaned across the scowling bride, and began to discuss mathematics. It wasn't long before the wife was in the back of the bus and Bergman next to his student — and so it remained for the rest of the bus trip! The story has a happy ending: the couple is still married, has a son who became a famous mathematician, and several grandchildren.

Presumably it was his preoccupation with mathematics that caused Bergman to appear to be out of touch with reality at times. For example, one day he went to the beach in northern California with a group of people, including a friend of mine who told me this yarn. Northem California beaches are cold, so when Bergman came out of the water, he decided that he'd better put on his street clothes. As he wandered off to the parking lot, his friends noticed that he was heading in the wrong direction; but they were used to his sort of behavior and paid him no mind. In a while, Bergman returned— clothed — exclaiming "You know, there's the most unfriendly woman in our car!"

Bergman was a prolific writer. Of course he worked in the days before the advent of word processors. His method of writing was this. First, he would write a manuscript in longhand and give it to the secretary. When she had it typed up, he would begin revising, stapling strips of paper over the portions that he wished to change. Strips would be stapled over strips, and then again and again, until parts of the manuscript would become so thick that the stapler could no longer penetrate. Then the manuscript would be returned to the secretary for a retype and the whole cycle would begin again.

Sometimes it would repeat ad infinitum, Bergman once told a student that "a mathematician's most important tool is the stapler." Bergman had a self-conscious sense of humor and a loud laugh. He once walked into a secretary's office and, while he spoke to her, inadvertently stood on her white glove that had fallen on the floor. After a bit she said "Professor Bergman, you're standing on my glove." He acted embarrassed and exclaimed "Oh, I thought it was a mousy." (It should be mentioned here that a number of wildly exaggerated versions of this story are in circulation, but I got this version from a primary source.)

Besicovitch

Abram S. Besicovitch (1891 — 1970) was a geometric analyst of extraordinary power. He became world-famous for his solution of the Kakeya needle problem. The problem was to find the planar region of least area with the property that a segment of unit length lying in the region can be moved through all direction angles θ, $0 \leq \theta \leq 2\pi$, within the region. Besicovitch's surprising answer was that for any $\epsilon > 0$, there is such a region with area less than ϵ.

Besicovitch, a Russian by birth, was a creature of the old world. After leaving Russia (a prudent move on account of his rumored black market dealings during World War I), Besicovitch ended up at Cambridge University in England. A dinner was given in his honor, at which the main course was some sort of game bird. In his thick Russian accent, Besicovitch asked the name of the tasty food that they were eating. *When he heard the reply, he exclaimed, "In Russia, we are not allowed to eat the peasants!"*

Besicovitch was a smart man, so he quickly became proficient at English. But it was never perfect. He adhered to the Russian paradigm of never using articles before nouns. One day, during his lecture, the class chuckled at his fractured English. Besicovitch turned to the audience and said "Gentlemen, there are fifty million Englishmen speak English you speak; there are two hundred million Russians speak English I speak". The chuckling ceased.

In another lecture series, on approximation theory, he announced "zere is no t in ze name Chebyshov." Two weeks later he said "Ve now introduce ze class of T-polynomials. Zey are called T-polynomials because T is ze first letter of ze name Chebyshov."

Besicovitch, in spite of his apparent powers, was modest. On his thirty-sixth birthday, he convinced himself that his best and most intense years of research were over. He said "I have had fourfifths of my life." Twenty-three years later, when in 1950 he was awarded the Rouse Ball Chair of Mathematics at Cambridge, someone reminded him of this frivolous remark. He replied, "Numerator was correct."

In the 1960s, the Mathematical Association of America made a series of delightful one-hour films in each of which a great mathematician gave a lecture, for a general mathematical audience, about one of his achievements. One of these films starred Besicovitch, and he explained his solution of the Kakeya needle problem. Besicovitch was a natty dresser under any circumstances, and he wore to this lecture an attractive light beige suit. However the lights were hot and, after a while he removed his jacket, revealing bright red suspenders! The producers were most surprised (this was thirty-five years ago, and nobody but firemen wore red suspenders), but the filming continued and the suspenders can be seen today.

At one point during the filming of Besicovitch, the aged professor had to blow his nose. He drew a large white handkerchief from his pocket and did so — loudly. Later, when Besicovitch viewed the finished product, he objected to the noseblowing scene as undignified — he wanted it removed. The producers were able to replace the offending video segment, but it was decided that the sound should remain. As a result, if you view the film today, there comes a point in the action where the camera abruptly leaves Besicovitch and focuses on the side of the room — and you can hear Besicovitch blow his nose.

Gödel

Kurt Gödel (1906 — 1978) was one of the most original mathematicians of the twentieth century. Any thesaurus links "originality" with "eccentricity," and Gödel had his fair share of both. Toward the end of his life, Gödel became convinced that he was being poisoned, and he ended up starving himself to death. However, years before that, his peculiar point of view exhibited itself in other ways.

Einstein was Gödel's closest personal friend in Princeton. *For several years Einstein, Gödel, and Einstein's assistant Ernst Straus (who later became a well-known combinatorial theorist) would lunch together.* During lunch they discussed non-mathematical topics — frequently politics. One notable discussion took place the day after Douglas MacArthur was given a ticker-tape parade down Madison Avenue upon his return from Korea. Gödel came to lunch in an agitated state, insisting that the man in the picture on the front page of the New York Times was not MacArthur but an imposter.

The proof? Gödel had an earlier photo of MacArthur and a ruler. He compared the ratio of the length of the nose to the distance from the tip of the nose to the point of the chin in each picture. These were different: Q.E.D.

Gödel spent a significant part of his career trying to decide whether the Continuum Hypothesis (CH) is independent of the Axiom of Choice (AC). In the early 1960s, a brash, young, and extremely brilliant Fourier analyst (student of the aforementioned Zygmund) named Paul J. Cohen (people who knew him in high school and college assure me that he was always brash and brilliant) chatted with a group of colleagues at Stanford about whether he would become more famous by solving a certain Hilbert problem or by proving that CH is independent of AC. This (informal) committee decided that the latter problem was the ticket. [To be fair, Cohen had been interested in logic and recursive functions for several years; *he may have conducted this seance just for fun.*] Cohen went off and learned the necessary logic and, in less than a year, had proved the independence. This is certainly one of the most amazing intellectual achievements of the twentieth century. Cohen's technique of "forcing" has become a major tool of modern logic, and Cohen was awarded the Fields Medal for the work. But there is more.

Proof in hand, Cohen flew off to the Institute for Advanced Study to have his result checked by Kurt Gödel. Gödel was naturally skeptical, as Cohen was not the first person to claim to have solved the problem, and Cohen was not even a logician! Gödel was also, at this time, beginning his phobic period. When Cohen went to Gödel's house and knocked on the door, it was opened six inches and a hoary hand snatched the manuscript and slammed the door. Perplexed, Cohen departed. However, two days later Cohen received an invitation for tea at Gödel's home. His proof was correct: The master had certified it.

Lefschetz

The story goes that Solomon Lefschetz (1884 – 1972) was trained to be an engineer. This was in the days, near the turn of the century, when engineering was part carpentry, part alchemy, and part luck (the pre-von Karman era). In any event, Lefschetz had the misfortune to lose both his hands in a laboratory accident. This mishap was lucky for us, for he subsequently, at the age of thirty-six, became a mathematician.

Lefschetz had two prostheses in place of his hands — they looked like hands, loosely clenched, but they did not move or function in any way. Over each he wore a shiny black glove. A friend of mine was a graduate student of Lefschetz; he tells me that one of his daily duties was to push a piece of chalk into Lefschetz's hand each morning and to remove it at the end of the day.

Lefschetz starred in one of the MAA films. He gave a lovely lecture, punctuated by a cacophony of squeaky chalk, about his celebrated fixed point theorem. His feelings about the film were mixed. At one point he says on film "I hope this is clear; it's probably about as clear as mud." After his lecture comes a filmed round table discussion including John Moore, Lefschetz, and a few others. For ten or fifteen minutes they reminisce about the old days at Princeton. One person reminds Lefschetz that in the late 1940s, during the heyday of the development of algebraic topology, they were on a train together. Lefschetz was asked the difference between algebra and topology. He is reported to have said "If it's just turning the crank, it's algebra; but if there is an idea present, then it's topology." *When Lefschetz was reminded of this story in the film, he became most embarrassed and said "I couldn't have said anything like that."*

With his artificial hands, Lefschetz could not operate a doorknob, so his office door was equipped with a lever. Presumably he had difficulty with other routine daily matters, too — dialing a phone, turning on a light, etc. By the time I was a graduate student at Princeton, Lefschetz was 87. He was still mathematically sharp but he had trouble getting around. In those days Fine Hall, the mathematics building in Princeton, was having constant trouble with the elevators: Push the button for the fifth floor and you're shot to the penthouse, down to the basement, and ejected on seven; or variations on that theme. The receptionist kept a log of complaints so that she could report them to the person who came to repair the elevator. One day Lefschetz got into the elevator and it delivered him to the fourth floor "machine room"; this room houses the air conditioning equipment and is ordinarily only accessible with a janitor's key. *Poor Lefschetz unwittingly wandered out into the room, only to have the elevator door shut behind him before he realized what was going on.* He was trapped in total darkness, could not summon the elevator (no key), could not turn the doorknob to use the stairwell, and could not find a telephone (which, even had he found, he probably could not have dialed). The members of the mathematics department rode that elevator for several hours, not realizing that Lefschetz was missing, before someone finally heard Lefschetz's shouts and understood what was going on. *Fortunately Lefschetz survived the incident unharmed.*

Speaking of the elevators at Princeton, one of my earliest memories as a graduate student was of the elevator emergency stop alarm going off three or four times a day. Especially puzzling was that everyone ignored it. Bear in mind that this alarm only sounds if someone inside the elevator sets it off. It is sometimes used by janitors to hold the elevator at a certain floor; but the janitors never used it during the day. After I had asked around for some time, someone finally told me the secret. When the mathematics department moved from old Fine Hall to new Fine Hall (sometimes called "Finer Hall," overlooking "Steenrod Square"), Ralph Fox, the famous topologist, was annoyed that there was no men's room on his floor. So, whenever he had to use the facilities, he would take the elevator to the next floor, set the emergency stop alarm, do what needed to be done, and then return to his floor. Now I knew why everyone smiled when the alarm went off. So much for boyhood memories; back to Lefschetz.

Lefschetz was famous for his aggressive self-confidence. He could terrorize most other mathematicians easily. At committee meetings he would pound his fist on the table with terrifying effect. *So it is with pleasurable surprise that one hears of exceptions.* The one I have in mind is a certain unflappable graduate student at the time of the student's qualifying examination. The qualifying exams at Princeton are administered as one long oral exam: three professors and one graduate student locked in an office for about three and one-half hours. The student is examined on real analysis, complex analysis, algebra, and two advanced topics of the student's choosing (*subject to the approval of the Director of Graduate Studies*). Our confident student had Lefschetz on his committee. Lefschetz was famous for, among other things, profound generalizations of Picard's theorems in function theory to several complex variables. So it came as no surprise when Lefschetz asked the student "Can you prove Picard's Great Theorem?" Came the reply "No, can you?" Lefschetz had to admit that he could not remember, and the exam moved on to another topic.

Lefschetz was one of those mathematicians, of whom we all know at least one, who would sleep during lectures and then wake up at the end with a brilliant question. At one colloquium, the speaker got stuck on a point about twenty minutes into his talk. A silence of several minutes ensued. This threw off Lefschetz's rhythm. He woke up, said "Are there any questions? Thank you very much," and the seminar was ended with a round of applause.

The "roasting" of an individual is a peculiarly American custom. A group of close friends holds a fancy dinner in honor of the victim, after which they stand up one by one and make a collection of (humorously delivered) insulting remarks about him. Some anecdotes are in the nature of a roast. Here is an example. In the fifties, it was said in Princeton that there were four definitions of the word "obvious." If something is obvious in the sense of Beckenbach, then it is true and you can see it immediately, If something is obvious in the sense of Chevalley, then it is true and it will take you several weeks to see it. If something is obvious in the sense of Bochner, then it is false and it will take you several weeks to see it. If something is obvious in the sense of Lefschetz, then it is false and you can see it immediately.

This last item reminds me of the old concept of "true in the sense of Henri Cartan." In the 1930s and 1940s, a theorem was "true in the sense of Cartan" if Grauert could not find a counter-example in the space of an hour.

The discussion of "truth" and "obviousness" raises the issue of standards. *Perhaps the least delightful arena in which we all wrestle with standards is that of referees' reports.* The Annals of Mathematics, Princeton's journal, has very high standards and exhorts its referees to be toughminded. Lefschetz was instrumental in establishing the pre-eminence of the Annals. But I doubt that even he could have anticipated the following event. Many years ago, Gerhard Hochschild (who sets high standards for himself and everyone else) submitted a paper to the Annals. The referee's report said "Good enough for the Annals. Not good enough for Hochschild. Rejected."

Wiener

The brilliant analyst Norbert Wiener (1894 — 1964) is a favorite subject of anecdotes. *He is just modern enough that many living mathematicians knew him and was just eccentric enough to be a neverending object of stories and pranks.*

Born the son of a distinguished professor of languages, Wiener became one of America's first internationally recognized mathematicians. Because of anti-Semitism in the American mathematical establishment, Wiener spent the early years of his career working in England. The story goes that when he met Littlewood, he said, "Oh, so you really exist, I thought that 'Littlewood,' was just a pseudonym that Hardy put on his weaker papers." Poor Wiener was so chagrined by this story that he denied it vehemently in his autobiography, thus inadvertently fueling belief in its validity. [In fairness to Wiener I should point out that another popular version of the story involves Edmund Landau: Landau so doubted the existence of Littlewood that he made a special trip to Great Britain to see the man with his own eyes.]

After Wiener left Britain, he moved to MIT where he stayed for more than twenty-five years. *He developed a reputation all over campus as a brilliant scientist and a bit of a character.* He was always working — either thinking or writing or reading. When he walked the halls of MIT, he invariably read a book, running his finger along the wall to keep track of where he was going. One day, engaged in this activity, Wiener passed a classroom where a class was in session. It was a hot day and the door had been left open. But of course Wiener was unaware of these details — he followed his finger through the door, into the classroom, around the walls (right past the lecturer) and out the door again.

People who knew Wiener tell me — and this comes through clearly in his autobiography as well— that he struggled all his life with feelings of inferiority. These feelings applied to non-mathematical as well as to mathematical activities. Thus, when he played bridge at lunch with a group of friends, he would invariably say, every time he bid or played, "Did I do the right thing? Was that a good play?" His partner, Norman Levinson, would patiently reassure him each time that he couldn't have done any better.

It is not a well-known fact that Wiener wrote a novel. The villain in the novel was a thinly disguised version of R. Courant. The hero was a thinly disguised version of Wiener himself. Friends were successful in discouraging him from publication. [Another version of the story is that the villain was Osgood. In the book, he proves a theorem that is a thinly disguised version of a celebrated theorem of Osgood, but in a different branch of mathematics; he ends up dying in China.]

Students liked to play pranks on Wiener. He read the newspaper every day at the same time in a certain lounge at MIT. As Wiener sat with the newspaper spread open before him, a student would sneak up and set the bottom of the paper afire. The results were spectacular, and the joke was repeated again and again.

And sometimes Wiener played jokes on his students, though he did not realize that he was doing so. On one occasion, a student asked him how to solve a certain problem. Wiener thought for a moment and wrote down the answer. The student hadn't really wanted the answer but wanted the method to be explained (this really was a long time ago!). So he said "But isn't there some other way?" Wiener thought for another moment, smiled, and said "Yes there is" — and he wrote down the answer a second time.

Probably the most famous Wiener story concerns a day when the Wiener family was moving to a new home. Wiener's wife knew Norbert only too well. So on the night before, as well as the morning of, the moving day, she reminded him over and over that they were moving. She wrote the new address for him on a slip of paper (the new house was just a few blocks away), gave him the new keys, and took away his old keys. Wiener dutifully put the new address and keys into his pocket and left for work. During the course of the day, Wiener's thoughts were elsewhere. At one point somebody asked him a mathematical question, and Wiener gave him the answer on the back of the slip of paper his wife had given him. So much for the new address! At the end of the day Wiener, as was his habit, walked home — to his old house. He was puzzled to find nobody home. Looking through the window, he could see no furnishings. Panic took over when he discovered that his key would not fit the lock Wild-eyed, he began alternately to bang on the door and to run around in the yard, Then he spotted a child coming down the street. He ran up to her and cried "Little girl, I'm very upset. My family has disappeared and my key won't fit in the lock." She replied, "Yes, daddy, Mommy sent me for you."

My final Wiener story, indeed my final story, does not seem to be well known, Even inveterate Wienerologists proclaim it too good to be true. But it's not too good for this article. I believe that I heard it when I was a graduate student at Princeton. As I've mentioned, Wiener was quite a celebrated figure on the MIT campus. Therefore, when one of his students spied Wiener in the post office,, the student wanted to introduce himself to the famous professor. After all, how many MIT students could say that they had actually shaken the hand of Norbert Wiener? However, the student wasn't sure how to approach the man. The problem was aggravated by the fact that Wiener was pacing back and forth, deeply lost in thought. *Were the student to interrupt Wiener, who knows what profound idea might be lost?* Still, the student screwed up his courage and approached the great man. "Good morning, Professor Wiener," he said. The professor looked up, struck his forehead, and said "That's it: Wiener!"

Exercises

1. *In the text, translate the sentences italicized.*

2. *Insert prepostions where necessary:*

to distinguish smth ... smth, smth is related ... smth, to be a witness ... smth, responsibility ... smth, to arrive ... the United

States, to arrive ... the Victoria Station, to arrive ... home, a conference ... several complex variables, lectures ... biholomorphic mappings, ... behalf of smb, to remind ... smth, to put ... clothes, to become famous ... smth, ... spite of smth, ... certain circumstances, to object ... smth, independent ... smth, ... the age of thirty six, to approach ... smth, to adhere ... smth.

3. *Using your English-English dictionary, check the pronunciation and stress of the words below:*

dozen, record (v.,n.), merely, renaissance, appreciate, poignant, scarce, graduate (v.,n.), aisle, processor, ad infinitum, conscious, exaggerate, source, circumstance, politics, thesaurus, exhibit, close (v.,adj.), hypothesis, colleague, comittee, latter, technique, alchemy, variable, issue, wrestle, doubt, pseudonym, vehemently, autobiography, inferiority.

4. *Give the English equivalents of:*

1) раз и навсегда
2) отнюдь не достаточный
3) человек, о котором идет речь
4) крупная конференция
5) прикладывать усилия
6) посещать лекции
7) аспирант (брит., амер.)
8) не обращать внимания
9) текстовый редактор
10) задача заключалось в том, чтобы найти
11) прочитать лекцию
12) при определенных обстоятельствах
13) что и требовалось доказать
14) найти контрпример
15) он завоевал репутацию
16) подшучивать над студентами

5. *Read the quotations below and answer the questions.*

Anonymous

A physicist learns more and more about less and less, until he knows everything about nothing; whereas a philosopher learns less and less about more and more, until he knows nothing about everything.

1. Explain what the author implies by nothing.

2. Could you agree with the second part of the quotation?

Winston Leonard Spencer Churchill (1874-1965), British author and Prime Minister.

Praise up the humanities, my boy. That will make them think that you are broad-minded.

3. Is it possible that Churchill spoke essentially about the same as the author of the quotation above?

4. Do you think Churchill considered the humanities to be inferior to the sciences?

David Zeaman (1921 -), American psychologist.

One of the differences between the natural and the social sciences is that in the natural sciences, each succeeding generation stands on the shoulders of those that have gone before, while in the social sciences, each generation steps in the faces of its predecessors.

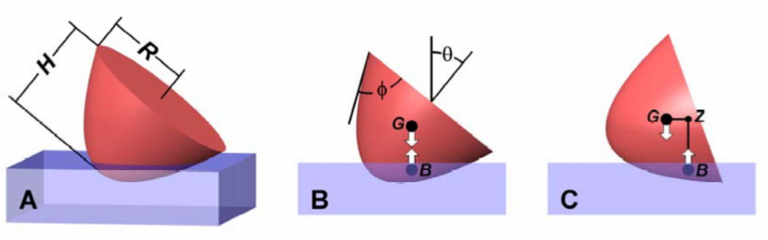
5. Do you think Zeaman meant that it is an intrinsic property of the social sciences when "succeeding generation stands on the shoulders of those that have gone before, while in the social sciences, each generation steps in the faces of its predecessors". Give examples. Give counter-examples.

XVIII. Completing Book II of Archimedes's *On Floating Bodies*

Chris Rorres, The Mathematical Intelligencer, volume 26, number 3, summer 2004, pp. 32-42.

Archimedes (c. 287 B.C. to 212/211 B.C.) lived in the Greek city-state of Syracuse, Sicily, up to the time that it was conquered by the Romans, a conquest that led to his death. Of his works that survive, the second of his two books of *On Floating Bodies*[1] is considered his most mature work, commonly described as a *tour de force*. This book contains a detailed investigation of the stable equilibrium positions of floating right paraboloids[2] of various shapes and relative densities, but restricted to the case when the base of the paraboloid lies either entirely above or entirely below the fluid surface.

This paper summarizes the results of research in which I completed Archimedes's investigation to include also the more complex cases when the base of the floating paraboloid is partially submerged. Modern scientific computing and computer graphics enabled me to construct a three-dimensional surface that summarizes all possible equilibrium positions (both stable and unstable) for all possible shapes and relative densities. This



equilibrium surface contains folds and cusps that explain certain catastrophic phenomena — for example, the sudden tumbling of a melting iceberg or the toppling of a tall structure due to liquefaction of the ground beneath it — that have long been observed but not previously explained fully.

Books I and II

Book I of *On Floating Bodies* begins with a derivation of Archimedes's Law of Buoyancy from more fundamental principles and finishes with a simple, elegant geometric proof that a floating segment of a homogeneous solid sphere is always in stable equilibrium when its base is parallel to the surface of the fluid, either above the fluid surface or below it. Book I introduced the concept of fluid pressure and initiated the science of hydrostatics. It took almost eighteen centuries before this work on the nature of fluids was continued by such scientists as Simon Stevin (Dutch, 1548-1620), Galileo Galilei (Italian, 1564-1642), Evangelista Torricelli (Italian, 1608-1647), Blaise Pascal (French, 1623-1662), and Isaac Newton (English, 1642-1727). In the interim, Book I served mainly as the basis for determining the density of objects, such as gemstones and precious-metal artifacts, by comparing their weights in air and in water.

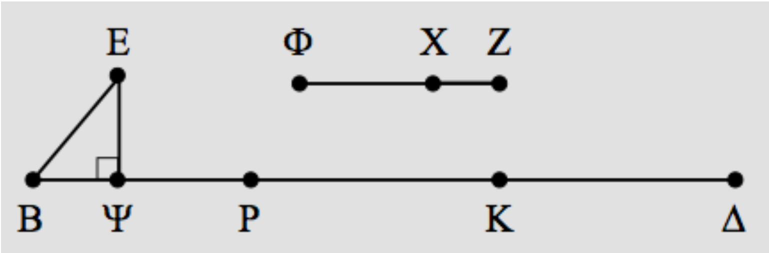
In Book II Archimedes extended his stability analysis of floating bodies from a segment of a sphere to a right paraboloid. However, Book II contained many sophisticated ideas and complex geometric constructions and did not have the appeal of Book I. *Only after Greek geometry was augmented with algebra, trigonometry, and* *analytical geometry and the field of mechanics reached the maturity to handle the concepts of equilibrium and stability that Archimedes introduced was Book II seriously studied.* It then became the standard starting point for scientists and naval architects examining the stability of ships and other floating bodies[3].

To describe the results Archimedes obtained in Book II let us first precisely define his object of study:

Definition: A *paraboloid* is a homogeneous solid convex object bounded by a surface obtained by rotating a parabola about its axis of symmetry and by a plane that is not parallel to the parabola's axis of symmetry. If the plane is perpendicular to the axis of symmetry it is called a *right paraboloid*, otherwise it is called an *oblique paraboloid*. The planar portion of the surface, which is either circular or elliptical, is called the *base* of the paraboloid.

Let R be the radius of the base of a right paraboloid and let H be its height (Fig. 1A). Define its *base angle* ϕ as the angle between 0^o and 90^o for which $\tan\phi = 2H/R$. In a profile view of the paraboloid it is the angle between its base and the tangent line to the parabolic cross section at the base (Fig. 1B). This base angle determines the shape of the parabola. Next, let ρ_{body} be the mass-density of the paraboloid and let ρ_{fluid} be the mass-density of the fluid in which it is floating within a uniform gravitational field. Following Archimedes, let us neglect the density of the air above the fluid[4] and define the *relative density* (or specific gravity) of the paraboloid as $s = \rho_{body}/\rho_{fluid}$, which is a number in the interval $[0, 1]$ for a floating paraboloid. Finally, let θ be the *tilt angle* (or *heel angle*), by which is meant the angle of inclination in the interval $[0^o, 180^o]$ of the axis of the paraboloid from the vertical with 0o corresponding to the base above the fluid level (Fig. 1B). As with Archimedes, let us confine the rotation of the paraboloid so that its axis always lies in a fixed vertical plane.

Below is an example of one of the ten propositions in Book II, in which I first give a very literal translation of the Greek text and then a very liberal modern translation. In the literal translation the 'axis' is a line segment whose length is the height H of the paraboloid and the 'line-up-to-the-axis' is the semilatus rectum of the paraboloid, which is a line segment of length $R^2/2H$. The last sentence in my



translation actually consists of seven excerpts from the beginning of the proof of Proposition 8 where a geometric construction is described.

Archimedes's Proposition 8. *Literal Translation:* A right segment of a right-angled conoid, when its axis is greater than one-and-a-half times the line-up-to-the-axis, but small enough so that its ratio to the line-up-to-the-axis is less than fifteen to four, and when further its weight has to that of the fluid [of equal volume] a ratio less than that which the square of the amount by which the axis exceeds one-and-a-half times the line-up-to-the-axis bears to the square of the axis, will, when so placed in the fluid that the base does not touch the surface of the fluid, not return to the vertical position and not remain in the inclined position except when its axis makes with the surface of the fluid a certain angle to be described.

[This angle is $EB\Psi$ in the diagram (Fig. 2) in which] (1) $B\Delta$ is equal to the axis; (2) BK is twice $K\Delta$; (3) KP is equal to the line-up-to-the-axis; (4) *the weight of the body is to that of the fluid [of equal volume] as the square of side ΦZ is to that of side $B\Delta$;* (5) ΦX is twice XZ; (6) ΦX is equal to $P\Psi$; and (7) the square of side ΨE is half of the rectangle of sides KP and $B\Psi$.

Archimedes's Proposition 8. *Modern Translation:*

A right paraboloid whose base angle ϕ satisfies $3 < \tan^2\phi < \frac{15}{2}$ and whose relative density s satisfies $s < (1 - 3\,\mathrm{ctg}^2\,\phi)^2$ has precisely one stable equilibrium position with its base completely above the fluid surface. The corresponding tilt angle is $\theta = \tan^{-1}\sqrt{\frac{2}{3}(1-\sqrt{s})\tan^2\phi - 2}$.

Archimedes's objective in Proposition 8 was to describe a geometric construction using compass and straightedge that begins with three lines segments describing the shape and relative density of the paraboloid (the axis, the line-up-to-the-axis, and the line segment ΦZ whose length is $\sqrt{s}H$) and ends with a diagram in which the tilt angle is revealed. My objective in the modern translation, however, was to summarize the geometric construction in a single analytical expression in which the equilibrium tilt angle θ is expressed as an explicit function of s and ϕ. *My modern translation incorporates centuries of algebraic, trigonometric, and analytical developments and considerably alters how the Greeks would have grasped Archimedes's results.* It also shows the limitations of Greek geometry in formulating and describing complicated physical phenomena.

Archimedes's other propositions in Book II complete his study of the stable equilibrium tilt angles when the base is either completely above or completely below the fluid surface for appropriate values of the base angle and the relative density. The main geometric tools he used were the formulas for the volumes and centroids of right and oblique paraboloids, formulas that he himself derived in other works[5]. The mechanical tools he used—again, tools that he himself first formulated — were his Law of Buoyancy for a floating body, his Law of the Lever, and the equilibrium condition that the center of gravity of the floating body must lie on the same vertical line as its center of buoyancy. (Because a paraboloid is a homogeneous convex body, its center of buoyancy coincides with the center of gravity of its submerged portion.)

Righting and Energy Arms

The numerical techniques I used required the evaluation of the moment acting on an unbalanced floating paraboloid. In Figure 1C a right paraboloid is floating in a fluid with the weight of the displaced fluid equal to the weight of the right paraboloid. However, it is not in equilibrium because the center of gravity G of the body is not on the same vertical line as its center of buoyancy B. Rather, the weight of the paraboloid and the buoyancy force form a couple that will cause the paraboloid to rotate in a counterclockwise direction toward the equilibrium position shown in Figure 1B. *The value of the couple, called the righting moment, is the weight of the paraboloid times the horizontal displacement GZ between G and B, taken as positive if B is to the right of G.* This horizontal displacement is called the *righting arm* and its use is preferred by naval architects to the righting moment. If a wave causes a ship to heel, the righting-arm expressed as a function of the heel angle affects the dynamics of how the ship will return to its vertical equilibrium orientation. One of the standard specifications of a ship is a graph of its righting arm for a wide range of heel angles.

If the base is completely above or below the fluid surface, it is possible to determine an exact expression for the righting arm of a floating right paraboloid using the exact formulas for the volume and centroid of an oblique paraboloid. For example, if the base is above the fluid surface then

$$\frac{\text{Righting Arm}}{H} = \frac{\sin\theta}{\tan^2\phi}\left[2 - \frac{2}{3}(1-\sqrt{s})\tan^2\phi + \tan^2\theta\right]$$

Setting this equal to zero determines all equilibrium tilt angles with the base above the fluid surface and, in particular, returns the expression for the tilt angle determined by Archimedes's Proposition 8 above. When the base is completely submerged, symmetry principles can be used to obtain an analogous expression[6].

While the righting arm provides the necessary information for the stability analysis of a floating body, its potential energy also provides some insight. Taking the fluid surface as the level of zero potential energy, the potential energy of the paraboloid/fluid system is the sum of the potential energy of the paraboloid and the potential energy of the displaced fluid. The potential energy of the paraboloid is its weight multiplied by the height of its center of gravity G above the fluid surface. Likewise, the potential energy of the displaced fluid is its weight (the same as the weight of the paraboloid) multiplied by the distance of its center of gravity B below the fluid surface. The total potential energy is then the weight of the paraboloid multiplied by the vertical distance between B and G. For a homogeneous convex paraboloid, G will always lie above B if the relative density is less than one and so the potential energy will always be positive.

By analogy with the term 'righting arm' I shall call the vertical displacement from B to G (BZ in Figure 1C) the *energy arm* of the floating paraboloid. The fundamental relationship between force and energy shows that when the righting arm and energy arm are expressed as functions of the tilt angle θ, then

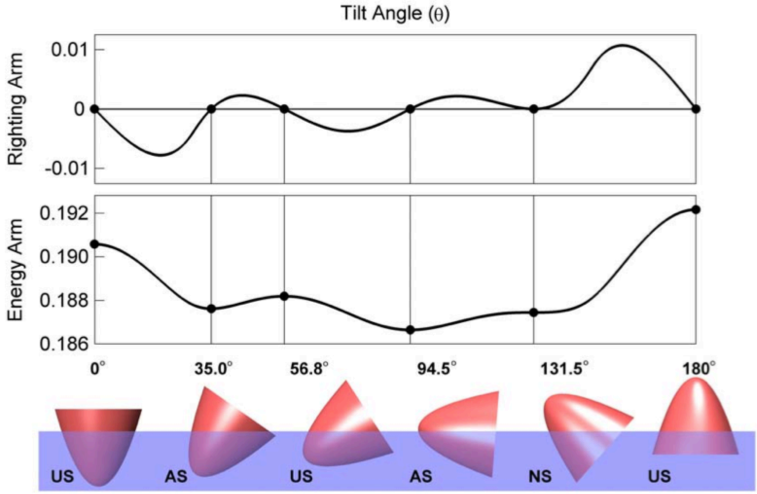
$$\frac{d(\text{energy arm})}{d\theta} = \text{righting arm.}$$

In order to work with dimensionless units, let us divide both the righting arm and the energy arm by the height H of the paraboloid. Thus one unit of the normalized energy arm is the energy needed to raise the paraboloid in air a distance equal to its height.

Figure 3 is an example of the normalized righting arm and the normalized energy arm as a function of the tilt angle for a right paraboloid with base angle 74.330^o and relative density 0.510. When its base is above the fluid surface ($0^o \leq \theta \leq 28.2^o$) I used Eq. (1) and when the base is below the fluid surface ($151.0^o \leq \theta \leq 180^o$) a similar exact expression was used.

When the base is cut by the fluid surface I used numerical integration to determine the volume and first moments of the unsubmerged portion of the paraboloid, from which the center of buoyancy and resulting righting-arm and righting-arm curves were determined[7]. The six roots of this righting-arm curve, or, equivalently, the six stationary points of the energy-arm curve, determine the six equilibrium positions of the corresponding paraboloid.

Because a positive righting arm produces a counterclockwise rotation and a negative righting arm produces a clockwise rotation, the way in which the algebraic sign changes through a root determines the stability classification of the corresponding equilibrium configuration. In particular, a root is *asymptotically stable* (AS), *neutrally stable to first order* (NS), or *unstable* (US) if the slope of the righting curve at the root is positive, zero, or negative, respectively[8]. *None of the six equilibrium positions for the particular paraboloid described in Figure 3 were present in*

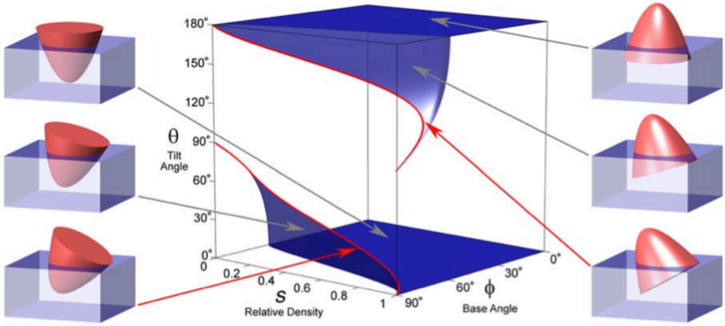


Archimedes's studies since they are either unstable or correspond to the base being cut by the fluid surface.

Archimedes's Results

Let us next summarize Archimedes's results in Book II in graphical form. In Figure 4 I have plotted a surface in (ϕ, s, θ)-space in the region $[0^o, 90^o] \times [0, 1] \times [0^o, 180^o]$ in which each point identifies a combination of base angle, relative density, and tilt angle for an AS equilibrium configuration of a paraboloid whose base in not cut by the fluid surface. The bottom portion of this equilibrium surface is associated with the base lying above the fluid surface and the top portion with the base lying below the fluid surface. Because of certain symmetry considerations[6] the top portion of the equilibrium surface is a rotation of its bottom portion about the line $s = 1/2$ and $\theta = 90^o$.

The curved piece of the bottom portion of the equilibrium surface, as partially determined by Archimedes's Proposition 8 above, has the explicit equation



$$\theta = \tan^{-1} \sqrt{\frac{2}{3}(1 - \sqrt{s}) \tan^2 \phi - 2}$$

restricted to the appropriate domain in ϕ and s. This curved surface is delineated below by its intersection with the plane $\theta = 0^o$ and this delineation identifies those configurations in which the paraboloid starts tilting from a vertical AS configuration. The curved surface is delineated above by the bottom red curve in Figure 4, which marks those configurations when the base of the paraboloid touches the fluid surface at precisely one point. In Proposition 10 of *On Floating Bodies II*, Archimedes developed a complicated geometric construction to determine these configurations. *His geometric construction is so ingenious as to warrant Cicero's assessment of him as being "endowed with greater genius that one would imagine it possible for a human being to possess".*

In modern analytical notation Archimedes's geometric construction for the bottom red curve is given by the following equations:

$$s = \left(\frac{6 + \tan^2 \theta}{6 + 5 \tan^2 \theta} \right)^4, \phi = \tan^{-1} \left(\frac{6 + 5 \tan^2 \theta}{4 \tan \theta} \right), 0^o \ leq \theta \leq 90^o,$$

where at $\theta = 90^o$ the limiting values $s = 1/625$ and $\phi = 90^o$ are taken.

For the upper surface corresponding equations can be obtained by replacing s by $1 - s$ by $180^o - \theta$ in Eqs. (3) and (4).

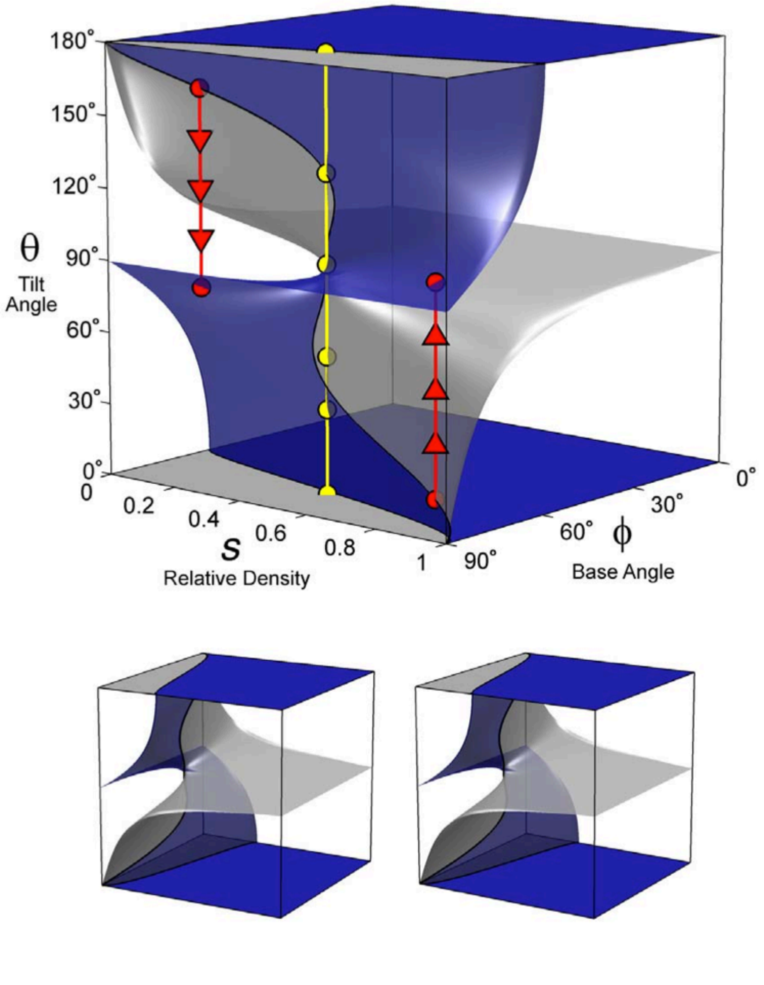
It should again be emphasized that Figure 4 and its analytical descriptions in Eqs. (3) and (4) are quite alien to Greek mathematics. As in the literal translation of Proposition 8, Archimedes could only express his complete results in convoluted sentences and complicated geometric constructions.

Complete Equilibrium Surface

My own research involved completing the equilibrium surface in Figure 4 by appending those points corresponding to AS configurations in which the base is cut by the fluid surface and also all points corresponding to US and NS configurations. The result is shown in Figure 5.

The construction of Figure 5 required determining all of the roots of the righting arm curves for a large number of base angles and relative densities using numerical techniques. *The base angle ϕ turned out to be a single-valued function of s in $[0, 1]$ and ϕ in $(0^o, 90^o)$. I used this fact, together with the rotational symmetry, to explicitly plot the surface. That is, rather than compute and plot θ as a multiple-valued function of ϕ and s, I computed and plotted ϕ as a single-valued function of s and θ for all s in [0, 1] and all θ other than $0^o, 90^o$, and 180^o.* For those three exceptional values of θ I used the facts that (1) the entire planes $\theta = 0^o$ and $\theta = 180^o$ are part of the equilibrium surface, indicating that the right paraboloid is always in equilibrium when its axis of symmetry is vertical, and (2) *the cross section of the equilibrium surface at $\theta = 90^o$ consists of the three line segments $\{ s = 1/2, \theta = 90^o \}$, where the paraboloid is on its side half in and half out of the fluid; $\{\phi = 0^o, \theta = 90^o\}$, where the paraboloid has collapsed to a circular disk; and $\{\phi = 90^o, \theta = 90^o\}$, where the paraboloid has collapsed to a line segment.*

The curved portion of the equilibrium surface resembles three-fourths of a turn of a helical surface, which is, appropriately enough, also the shape of an Archimedes screw. However, the axis of the helical surface is distorted. *It is about this distorted axis, near the*


180°
150°
120°
θ
90°
Tilt
Angle
60°
30°
0°
0
0.2
0.4
0.6
0.8
1
s
Relative Density
0°
30°
60°
90°
φ
Base Angle

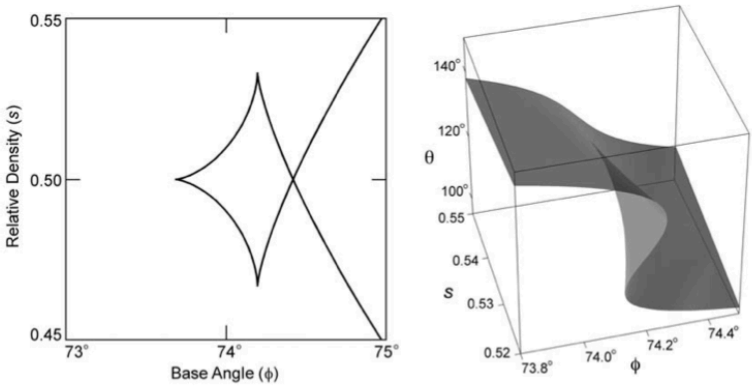
vertical line { $\phi = 74^o, s = 1/2$ }, that one finds up to seven distinct values of the tilt angle for fixed values of ϕ and s and a variety of complicated equilibria transitions.

The equilibrium surface is colored with the AS points in blue, the US points in gray, and the NS points in black. The NS points lie on one continuous curve that separates the equilibrium surface into AS and US pieces. On the plane $\theta = 0^o$ the curve of NS points has the equation $s = (13 \cot^2 \phi)^2$, which Archimedes had previously identified as the limiting condition for a vertical stable equilibrium (cf., Proposition 8). By symmetry, a similar curve lies on the plane $\theta = 180^o$.

To determine the stability of each nonvertical equilibrium, I determined the algebraic sign of the slope of the corresponding righting curve at the corresponding root. When the fluid level does not cut the base, Archimedes's results are applicable. When the base is cut by the fluid level, the algebraic sign can be determined by computing the ratio of the two principal moments of inertia of the cross section of the intersection of the paraboloid with the fluid surface. The cross section in this case is a right segment of an ellipse and I determined its principal moments of inertia using exact formulas.

For base angles of less than 60^o the right paraboloid has the same floating characteristics as the spherical segment that Archimedes studied in Book I: namely, for any relative density it floats stably at the vertical tilt angles 0^o and 180^o and unstably at a tilt angle close to 90^o. *I shall refer to this as plate-like behavior, in contrast to the rod-like behavior when the base angle of the paraboloid is close to 90^o. In the latter case the paraboloid floats unstably at 0^o and 180^o for most densities and floats stably at a tilt angle close to 90^o, when it is lying on its side.* Because plate-like and rod-like paraboloids float in totally different ways, the transition between the two shapes produces a complicated equilibrium surface with correspondingly complicated floating behaviors.

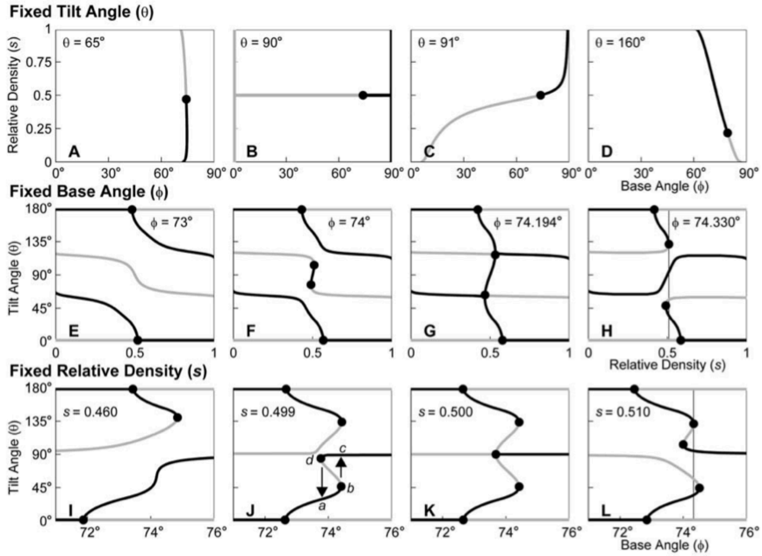
The NS points on the curved portion of the equilibrium surface lie along the edge of a helical fold that leads to catastrophic transitions between two equilibria as the base angle and/or the



relative density of the floating paraboloid changes. These NS points identify saddle-node bifurcations where an US point and an AS point meet and annihilate each other, forming a fold catastrophe. *If the parameters of a floating paraboloid change in such a way as to pass over a fold, the equilibrium configuration will jump catastrophically from the NS point on the fold to an AS point lying on the vertical line through the NS point.* The NS points on the curved portion of the equilibrium surface and the corresponding fold catastrophes arise only when the base of the paraboloid is partially submerged and so did not enter into Archimedes's consideration.

Cusp Catastrophes, Bifurcations, and Hysteresis Loops

Figure 6(left) is a projection of a portion of the curve of fold catastrophes onto the ϕs-plane. Although this curve is smooth in three-dimensional space, its two-dimensional projection has three cusps. These cusps identify three cusp catastrophes at (ϕ, s, θ)- values of $(74.19^o, 0.467, 60.0^o)$, $(73.68^o, 1/2, 90^o)$, and $(74.19^o, 0.533, 120.0^o)$. These are points where the equilibrium surface folds over and locally changes from a single- value function of θ to a triple-valued function. Figure 6(right) is an oblique view of the topmost cusp catastrophe illustrating this folding behavior.



Within the diamond-shaped region in Figure 6(left) outlined on the left by the three cusps, the equilibrium surface has seven tilt-angle values, including two US values of 0^o and 180^o.

Figure 7 contains twelve slices of the equilibrium surface for fixed values of the tilt angle, base angle, and relative density. These slices exhibit the complicated geometric nature of the equilibrium surface, which leads to complicated changes in the equilibrium position of the paraboloid as its base angle or relative density changes. Figures 7A to 7D illustrate the fact that is a single-valued function of s for all θ between 0^o and 180^o other than 90^o. Figures 7E to 7H exhibit pitchfork bifurcations at θ equal to 0^o and 180^o and show the bifurcations associated with the passing of the slice through the cusp catastrophe at a base angle of 73.682^o ($=$ $\tan^1 \sqrt{35/3}$) between Figures 7E and 7F. In Figure 7G the slice passes through the two cusp catastrophes at tilt angles of 60.0^o and 120.0^o producing two more pitchfork bifurcations.

Figures 7I to 7L show passages through the three cusp catastrophes using slices of constant relative density. Within the slice at $s = 1/2$ (Fig. 7K) the cross-section of the cusp at $\theta = 90^o$ appears as a subcritical pitchfork bifurcation.

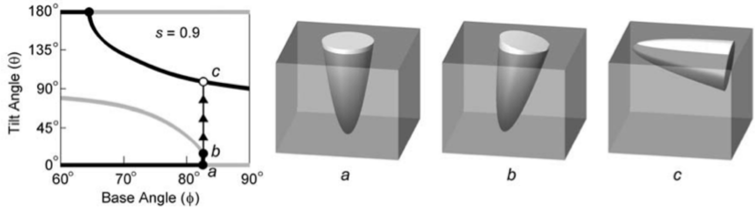
Small hysteresis loops appear in the s-slices for s between 0.467 and 0.500 associated with the cusp at $\theta = 60.0^o$ and for s between 0.500 and 0.533 associated with the cusp $\theta = 120.0^o$. Figure 7J highlights the loop for $s = 0.499$. The paraboloid flips catastrophically about this loop between the two orientations (a) and (c) in a periodic manner as the base angle oscillates between the values 73.8^o and 74.4^o, a change of only 0.6^o. As the base angle goes through one oscillation the tilt angle continuously increases from 29.0^o to 43.3^o (a to b), then jumps to 89.6^o (b to c), then decreases continuously to 86.6^o (c to d), and finally returns catastrophically to 29.0^o (d to a).

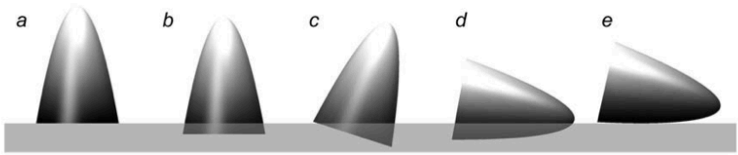
The vertical lines in Figures 7H and 7L are at $s = 0.510$ and $\phi = 74.330^o$, respectively, and pass through the six tilt angles shown in Figure 3. A slight increase in either the relative density or the base angle from these values cause the structurally unstable NS point at $\theta = 131.5^o$ to be annihilated, while a slight decrease causes it to split into an AS-US pair.

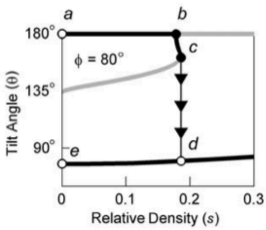
Tumbling of icebergs due to melting

Icebergs are notoriously unstable and may tumble over for no apparent reason. Jules Verne gave an explanation of this phenomenon in his 1870 novel *20,000 Leagues Under the Sea.* After a tumbling iceberg strikes the Nautilus, Captain Nemo explains, "An enormous block of ice; a mountain turned over. When icebergs are undermined by warmer waters or by repeated collisions, their center of gravity rises, with the result that they overturn completely".

Figure 8 quantifies this phenomenon for a paraboloidal iceberg with uniform relative density of 0.9 melting in such a way that its base angle slowly increases (i.e., it gets narrower[9]). The cross-section of the equilibrium surface at this relative density shows that for base angles less that 82.54o the iceberg can float stably in a vertical orientation with its base above water (a). As its base angle







slowly melts from 82.54^o to 82.65^o, its tilt angle slowly increases from 0^o to 12.3^o (a to b), and then suffers a catastrophic jump to 98.1^o when the base angle increases past 82.65^o (b to c).

The paraboloidal iceberg will tumble, rather than gradually roll over, only if its relative density is greater than 0.467 (Figures 8I and 8J). The tumbling then takes place almost immediately after the base cuts the fluid surface.

Toppling of structures due to soil liquefaction

During an earthquake loose, water-saturated soil can behave like a viscous fluid, a phenomenon known as soil liquefaction. Structures originally supported by the soil begin to float on it when it liquefies and can then sink and topple as the density of the liquefied soil decreases.

Figure 9 illustrates this phenomenon for a paraboloidal structure with a base angle of 80^o initially standing vertically on solid ground at a tilt angle of 180^o (a). Let us consider solid ground as a liquid with infinite density, so that the relative density of the structure is zero. As the ground liquefies its density slowly decreases from infinity through large finite values and the relative density of the structure increases from zero through small finite values. The cross-section of the equilibrium surface at a base angle of 80^o shows that as the relative density of the structure increases from 0 to 0.177 the structure slowly sinks into the ground in a vertical position (a to b), then starts to slowly tilt until it reaches a tilt angle of 162.6^o at a relative density of 0.187 (b to c), at which point its base is barely above ground. If the relative density increases further, the structure topples catastrophically to a tilt angle of 79.9^o (c to d). This toppling is irreversible. If the soil returns to its solid state, the structure, if still in one piece, ends up at a tilt angle of 77.2^o (d to e).

As with the iceberg, the paraboloidal structure cannot topple until its base is partially exposed above the soil level. Additionally, this toppling can only occur if the base angle of the structure is greater that 74.194^o (cf., Fig. 7G). For smaller base angles the paraboloidal structure gradually sinks and tilts into the soil without toppling as the soil's density decreases.

Conclusion

One needs only glance at Archimedes's Proposition 8 above to see that On Floating Bodies is several orders of magnitude more sophisticated than anything else found in ancient mathematics. It ranks with Newton's Principia Mathematica as a work in which basic physical laws are both formulated and accompanied by superb applications.

However, Archimedes's investigation of floating paraboloids had to await the computer age for its continuation, just as did his famous Cattle Problem. This latter problem has an integer solution with more than 200,000 digits that needed modern computers to determine. Likewise, I needed advanced computing and graphics systems to determine all possible equilibrium positions of Archimedes's floating paraboloids and to represent them in a single diagram.

No doubt Archimedes would have been interested in seeing the results in this paper, but one could ask how much of the mathematics developed in the last two millennia he would need to learn to understand them. At the very least he would have to learn about three-dimensional Cartesian coordinate systems, although he should have no trouble with this concept considering how close he came to defining a polar-coordinate system in his description of the spiral that bears his name. Unhooking him from the straitjacket of compass-and-straightedge construction to explain how the relationship among three variables can be represented by the points on a surface might take a little longer. He could then see how the equilibrium surface in Figure 5 presents a global picture of the behavior of his floating paraboloids and how the twists and turns of that surface lead to catastrophic transitions. He could also then appreciate some of the advances made in mathematics in the last 23 centuries, although *my guess is that he would have expected more considering the enormous advances that he alone made in his lifetime.*

Acknowledgments

My sincere thanks to Professor C. Henry Edwards of the University of Georgia and Prof. Dr. Horst Nowacki of the Technical University of Berlin for their advice and support.

Notes

1. A Greek manuscript dating from about the ninth century and containing both books of *On Floating Bodies* was translated into Latin by the Flemish Dominican William of Moerbeke in 1269, along with other works of Archimedes from other manuscripts. The tracks of the Greek manuscript were lost in the fourteenth century, but Moerbeke's holograph remains intact in the Vatican library (*Codex Ottobonianus Latinus* 1850). Moerbeke's Latin translation was the source of all versions of *On Floating Bodies* from his time until the twentieth century. Moerbeke's translation of both books of

On Floating Bodies was first printed in 1565, independently by Curtius Troianus in Venice and by Federigo Commandino in Bologna. A palimpsest from the tenth century, discovered and edited by J. L. Heiberg in 1906, contains the only extant Greek text. The texts by Dijksterhuis and Heath are the only translations/paraphrases presently available in English.

2. Also called *parabolic conoids* or *orthoconoids*.
3. *Some classic works concerned with how things float are: Christiaan Huygens (Dutch, 1629-1695), De iis quae liquido supernatant; Pierre Bouguer (French, 1698-1758), Traite du Navire, de sa Construction, et de ses Mouvements; Leonhard Euler (Swiss, 1707-1783), Scientia navalis; Jean Le Rond d'Alembert (French, 1717-1783), Traite de l'equilibre et du Mouvement des Fluide; Fredrik Henrik af Chapman (Swedish, 1721- 1808), Architectura Navalis Mercatoria; George Atwood (English, 1745- 1807); The Construction and Analysis of Geometrical Propositions Determining the Positions Assumed by Homogeneal Bodies Which Float Freely, and at Rest, on the Fluid's Surface; also Determining the Stability of Ships and of Other Floating Bodies; Pierre Dupin (French, 1784-1873), Applications de geometrie et de mecanique; August Yulevich Davidov (Russian, 1823-1885); The Theory of Equilibrium of Bodies Immersed in a Liquid* [in Russian]. More recent works include.
4. If air is the mass-density of the air, then, because the paraboloid is a homogeneous convex body, the buoyancy effect of the air can be accounted for by defining the relative density as $s = (\rho_{body} - \rho_{air})/(\rho_{fluid} - \rho_{air})$. Actually, Archimedes's description of s as the ratio of the weight of the body to the weight of an equal volume of fluid results in this expression if the weighing is done in air, but it is doubtful that he was aware of the buoyancy effects of air.
5. Archimedes's proof for the volume of a right or oblique paraboloid is contained in Propositions 21-22 of *On Conoids and Spheroids.* He gave a 'mechanical' proof of the location of the centroid of a right paraboloid in Proposition 5 of *The*

Method. He used the correct expression for the centroid of an oblique paraboloid in *On Floating Bodies II*, but no proof survives.

6. Symmetry considerations show that if θ is an equilibrium tilt angle when the relative density of a floating body of revolution is s, then $180^o - \theta$ is an equilibrium tilt angle for the body when its relative density is $1\breve{}s$. Thus only tilt angles in the range $[0^o, 90^o]$ need be explicitly computed. Although Archimedes does not mention this fact, it is clear that he was aware of it for his paraboloids since his proofs when the base is below the fluid surface are the same, *mutatis mutandis*, as his proofs when the base is above the fluid surface.
7. The integrals determining the volume and centroids of the unsubmerged portion can be found in closed form using symbolic algebra programs, but they are page-long monstrosities and numerical integration yields results much quicker and with more accuracy. Additionally, numerical techniques were used to determine when the weight of the displaced fluid is equal to the weight of the paraboloid and to find the roots of the righting arm curve. The symbolic calculations were performed with MapleTM and MathematicaTM and the numerical calculations and graphs were performed with MatLabTM.
8. Points NS to first order may be AS or US when higher-order terms are considered. In particular, the NS points when $\theta = 0^o$ and 180^o are actually AS and the rest are US. These NS points are also classified as *nonhyperbolic, degenerate*, and *structurally unstable* .
9. Unlike Verne's iceberg, the center of gravity of the paraboloidal iceberg remains fixed relative to its size at a distance of one-third of its height along its axis from its base.
10. In Greek: CATASTROPHE = $KATA\Sigma TPO\Phi H$ = a downward turn

Exercises

1. *In the text, translate the sentences italicized.*

2. *Insert prepostions and translate the following:*

to lead ... smth, due ... smth, parallel ... smth, ... the interim, to extend analysis ... smth ... smth, bounded ... smth, to rotate a parabola ... its axis, perpendicular ... smth, ... the interval, ... the line, ... the plane, ... Figure 3, ... particular, to multiply smth ... smth, to be equal ... smth, stable ... first order, to pass ... a point, to restrict smth ... smth, to replace smth ... smth, projection of smth ... smth, to date ... the ninth century, to translate ... one language ... another language, ratio of smth ... smth, to correspond ... smth.

3. *What do the abbreviations* **BC, c.** (e.g. *in c. 287 BC*)*,* **cf., i.e.** *stand for?*

4. *Give the English equivalents of the following:*

1) подробное исследование
2) (не)устойчивое положение равновесия
3) сложный случай
4) вывод закона плавучести
5) однородное тело
6) ввести понятие давления
7) определить вес вытесненной жидкости
8) точно определить цель исследования
9) касательная к кривой
10) однородное гравитационное поле
11) отрезок прямой
12) достаточно малый
13) меньше (больше), чем
14) значительно изменять
15) соответствующие значения угла
16) численные методы
17) вращать против часовой стрелки
18) приравнивать к нулю
19) в частности

20) по аналогии
21) безразмерные единицы
22) в силу симметрии
23) под углом 90^o к
24) необратимый процесс
25) декартова система координат
26) явление может быть объяснено с помощью
27) интегралы в явном виде можно найти
28) давать результаты с большей точностью
29) тело вращения
30) проводить вычисления с помощью

5. *Use your dictionary to check the pronunciation and stress of the words below:*

either, catastrophic, catastrophe, surface, buoyancy, hydrostatics, architect, oblique, height, tangent, ratio, determine, compass, infinite, lever, finite, technique, coordinate (v., n.), ingenious, helical, ellipse, characteristic, annihilate, exhibit, occur, doubt, guess, homogeneous, yield, hyperbolic, degenerate (adj.).

6. *What are the plural forms of the following nouns? Use a dictionary.*

Equilibrium, phenomenon, formula, axis, hysteresis, millenium.

7. *Make up your own sentences with the following words:*

up to the time, to carry out an investigation, to do research, to extend analysis from ... to, to let smth be smth, to define smth as smth, to make some angle with, our objective is to ... , respectively, to take place, to be concerned with.

8. *Read the quotations below and answer the questions.*

Aristotle (384-22 BC), Greek natural philosopher.

The same ideas, one must believe, recur in men's minds not once or twice but again and again.

1. Could you give any examples from the history of sciences?

Jean Baptiste Joseph Fourier (1768-1830), French mathematician and physicist.

Profound study of nature is the most fertile source of mathematical discoveries.

2. Could you say what Fourier's results may have originated from nature?

3. Give other examples of natural phenomena that led to profound mathematical results.

4. Give examples of the opposite situation when pure mathematics is applied to explain facts from nature.

XIX. Appendix

I. Poems

Math Class Limericks

Marion Cohen, The Mathematical Intelligencer, volume 21, number 4, Winter 1999, p. 3.

We are algebra-shy nincompoops.
We can't get a grip on our groups.
We ask in a frenzy,
Is it $\mathbb{Z}_n$ or $n\mathbb{Z}$?
We guess wrong, so we grin and go Oooops.

(God to Kronecker)
For seven long days labored I
making integers, low and then high.
But now 'tis day eight.
It is time to create
the fractions and square roots and π.

(God's last word to Kronecker)
I couldn't see stopping at 10.
Then I couldn't see stopping at n.

Then $\mathbb{Z}$ and then $\mathbb{Q}$
and the square root of 2,
but I'm stopping with C. Amen.

Ode to Andrew Wiles, KBE

By Tom M. Apostol, The Mathematical Intelligencer, volume 22, number 4, Fall 2000, p. 36.

Fermat's famous scribble - as marginal note -
Launched thousands of efforts - too many to quote.
Anyone armed with a few facts mathematical
Can settle the problem when it's only quadratical.
Pythagoras gets credit as first to produce
The theorem on the square of the hypotenuse.

Euler's attempts to take care of the cubics
Might have had more success if devoted to Rubik's.
Sophie Germain then entered the race
With a handful of primes that were in the first case.

Lame at mid-century proudly announced
That the Fermat problem was finally trounced.
But the very same year a letter from Kummer
Revealed the attempt by Lame was a bummer.

Regular primes and Kummer's ideals
Brought new momentum to fast-spinning wheels.
Huge prizes were offered, and many shed tears
When a thousand false proofs appeared in four years.
The high-speed computers tried more and more samples,
But no one could find any counterexamples.

In June '93 Andrew Wiles laid claim
To a proof that would bring him fortune and fame.
But, alas, it was flawed - he seemed to be stuck -
When new inspiration suddenly struck.

The flaw was removed with a change of approach,
And now his new proof is beyond all reproach.
The Queen of England has dubbed him a Knight
For being the first to show Fermat was right.

The Königsberg Bridge Poem

Homage to Euler

By Judith Saunders, The Mathematical Intelligencer, volume 19, number 4, Fall 1997, p. 20.

Flowing through Königsberg and spanned by seven bridges, the River Pregel surrounds an island (called Kneiphof) in the middle of the city. It is said that people used to entertain themselves by trying to devise a route around Königsberg which would cross each bridge just once.

Like mice in mazes, locals scampered forth
and back, around and through the town, traversed
and re-traversed the central island, bent
on crossing each of seven bridges once
and only once.
You alone declined
to join the briskly questing citizenry.
Knowing the elusive route would not
be found on foot, in chance meandering
(some providential Spaziergang)
you proceeded without stirring from
your chair to take a different sort of trip.

With pen and paper first you razed the place,
demolished houses, Marktplatz, terraces
and domes. You rid it of shrubbery and trees.
Walls fell. Cathedrals crumbled. Squirrels, ducks
and hedgehogs vanished. Not a lamppost was spared,
not a Denkmal stood. No cobblestone escaped
your ruthlessly obliterating hand.

And when you'd sheared away particulars,
trimmed Königsberg to the bone, you saw
a skein of penstrokes, luminous patterns
of points and lines, necessary sequences:
where trails of connectivity led, and where
they failed, and why – no matter time or place,
terrain or weather.
Ineluctably
you built and crossed a single, Ideal Bridge
to reach a quiet Kneiphof of the mind,
an island of essences. Stripped stark. Clean.
Bare bedrock of a new geometry.

V(n) = V(n-V(n-1)) + V(n-V(n-4))

Kelie O'Connor Gutman, The Mathematical Intelligencer, volume 23, number 3, summer 2001, pp. 50

Recalling a Collaboration with Greg Huber and Doug Hofstadter

And now, my friends, in poetry,
The lowdown on the function V,
Which calls itself recursively.
My verse will mirror it, you'll see.
The code pertains to how it rhymes
In trios, couplets, singletons -
But that we'll save until the end.
Let's start with all those dense parens
And minus-signs and V's and n's
That make my title hard to sing.
V's formula (which yields a string
Akin to Fibonacci's sequence)
Says "Add two prior values found
To get the next - thus, round and round,"
But V demands that one look back
To distant spots along its track.
To find those places that one visits,

Step back by one and back by four -
But do not add these, I implore,
As Fibonacci might; there's more.
These merely serve as indices
For two more countbacks, if you please,
That yield two summands for your summing.
A short example would be great,
So here's V's startup - one through eight:
1, 1, 1, 1, 2, 3, 4, 5;
It's V(9) we'll now derive.
All set? Let's get those brain cells buzzing!
Replace the n's by 9; subtract
To get first 8, then 5. Extract
V's values at those spots exact.
Thus V at 5 delivers 2,
Whilst V at 8 gives 5. Now you
Must do a wee bit more subtracting.
So take these indices from 9 -
Get 4 and 7. These define
The spots in V that we must add.
Take V(4) - this won't be bad - Our table tells us it's a 1,
While V(7)'s 4 - well done!
We're nearly finished with our run,
For V(9) (towards which we strive)
Is 4 + 1 (their sum) - thus 5.
I'm sure that wasn't too demanding.
A question now to contemplate:
What makes this function captivate
The few who've tarried in its thrall?
Well, first of all, from something small,
A sequence starts that never ends.
Surprisingly, as V extends,
It hits each number in succession
And never ever skips a beat
While marching up its one-way street,
With no looks back and no retreat.
A visit to each number's paid,

With ne'er the welcome overstayed.
Aside from four 1's at the start-up,
Each number's tapped three times at most,
And gets, as said, at least one toast.
V's charm lies in its wondrous mix
Of ordered chaos, as it clicks
Its way along the number line.
No pattern's clear in its design,
Yet hidden truths are there to mine.
A different way to look at V -
Through groups of length 1, 2, or 3 -
Involves observing repetitions.
If there's a value that's dunned thrice
Successively (or once, or twice),
We say a "clump" is there, size 3
(Or 1, or 2, respectively).
The list of clumps shows how V duns
The integers, with 3's, 2's, 1's.
Transitions give another viewpoint;
They show just how the numbers climb -
At most two jumps come at one time,
And then, plateau. It's quite sublime!
Thus, novel views of V's quaint bumps
We gain by listing climbs and clumps -
I.e., two complement'ry ways
With which upon V's path to gaze.
Oh, V both baffles and unites
Those few of us who've set our sights
On understanding its delights.
P.S. - For those who'd like to see
A longer stretch of sequence V,
Within this poem I've encrypted
The list of clumps for you to find.
I'll help at first, if you don't mind.
The first four rhymes - jot down a "4" ;
That digit you will see no more.
And next we have an unrhymed run,

For those three lines write "1, 1, 1".
The rest is readily decoded:
When three lines rhyme (like these), write "3" ;
Write "2" for couplets; finally,
Write "1" for rhymeless lines. Let's see:
4, 1, 1, 1, 2, 2, 1, 2. . . -
My poem's rhyme scheme. Now go through
The clumps, translating; I'll assist.
4 ones, 1 two, 1 three. . . - the list
That's on line twenty-six! This sequence
In fact gives back V's funky grace,
A lovely gem in function space.

II. A Deprogrammer's Tale

A Deprogrammer's Tale

Colin Adams, The Mathematical Intelligencer, Volume 23, No. 4 (Fall 2001), pp. 13-14.

They hooked him in calculus class. Started slow. Didn't want to be too obvious. Gave him a little trig review, some functional notation, and then introduced limits. Gave him lots of problems to work. Kept him busy to get his guard down. Then pow, hit him with the concept of the derivative. The raw power and simplicity of the idea, it was overwhelming. How could he resist? Who can? I know. I've been through it myself. Yes, that's right. I was one of them once. I was a slave to mathematics.

But unlike most, I escaped. And now my life is dedicated to helping others who were not as fortunate as I.

In this particular case, I was hired by the parents of one Lawrence Desenex. One minute, Larry was pre-med, heading for a lucrative plastic surgery practice in Cherry Hill, and the next minute he was talking about earning a Ph.D. in mathematics. All thought of financial gain went out the window. His parents were horrified. Dreams of my-son-the-doctor turned into nightmares of my-son-the-itinerant-mathematician. But me, I wasn't surprised when I heard the tale. I'd heard it a hundred times before. Believe it or not, 1200 people a year get Ph.D.s in math in the United States alone. That sounds incredible, but I understand why; I know the seductive power of a beautiful proof, the appeal of a well-turned lemma. Larry had fallen prey in the usual manner. After hearing the derivative explained in a lecture hall with 300 other students, he went to see the professor during office hours. That's when they know they have you. You're one of the susceptible ones, looking for some meaning beyond the plug and chug problems.

A little chitchat, maybe notational, a bit of history, Newton verses Leibnitz, that sort of thing, all seemingly innocuous. And then, when he least expected it, the epsilon delta definition of a continuous function. Poor guy was putty in the professor's hands. Before he could get his head back on straight, the professor invited him to a departmental colloquium, followed by tea. Larry dutifully went, and although he was blown out of the water by the material, he saw the others there, at rapt attention, and he felt he was among friends.

At tea, the department members ignored Larry, feigning indifference to the freshman who was interested in math, pretending they were too wrapped up in their own research to care. But oh, if he only knew. They were watching his every move, as they scribbled on the blackboard and talked about this theorem or that with their colleagues. He was a marked man, and Larry didn't even know it.

In cases like these, there is a small window of opportunity, a short period when a student can yet be saved. But you must act fast. Once students take Real Analysis and Abstract Algebra, their fate is sealed. The window has been slammed shut and shuttered.

But Larry's parents had called me in time. He was taking Linear Algebra, the applied version. There was hope yet.

I found him in the cafeteria with an untouched plate of tuna casserole and a copy of The Man Who Loved Only Numbers open in front of him. I gave him my winningest smile.

"Erdös, huh? Mind if I join you?"

He was clearly impressed and motioned to the seat across the table.

"Like math, do you?" , I asked.

"Oh, yes," he said enthusiastically. "It's so beautiful."

"Yes, it does have an appeal."

"Have you ever seen the argument for the uncountability of the reals?", he asked. "That's really cool." The bubbly excitement, the glassy bright eyes. Oh, he was in deep. We talked math for a while. I played along. Euclid this, Euler that. Then I laid the trap.

"Hey, my roommate and I are having a birthday celebration for Karl Friedrich Gauss on Wednesday at my apartment. You're invited."

Of course, he was thrilled. Susceptible and trusting are two descriptions of the same attribute.

He showed up right on time. It hadn't taken him long to pick up that characteristic of mathematicians. I let him in and locked the door behind him. Then everyone popped out, his parents, his grandparents, a cousin, an aunt, his best friend from high school.

"What's going on here?" he said, clearly at a loss. "This isn't a birthday party for Gauss."

"No, it's not," I said. "Gauss was born in April. This is an intervention, Larry. These are the people who love you and they're here to help." He backed away.

"Open the door. Let me go," he cried desperately. I blocked him.

"Not until you hear what we have to say."

He looked like Galois after the duel. The blood drained from his face. Must have been wondering where his muse was now.

His mother spoke first. "Bunchkins, bunchkins, have you thought about us? We love you, Pinchy, but good gracious, what would the neighbors say? Mrs. Krawlick would revel in the news. Our son, a mathema, a mathema. . . , I can't say the word."

She began to bawl uncontrollably. Larry's father held her. "Look at your mother. Look at what you are doing to her. She can't even say the word."

"Poor, poor Erma," said his aunt, patting Larry's mother on the sleeve. "Larry, I can't believe you would do this. You seemed like you were a good kid. You used to watch television. You had a lemonade stand. What happened to you? My kids would never do this. Evan here, now, he is a dentist, aren't you Evan?"

The cousin nodded yes.

"And Cybil works in marketing for an ad agency. And I am proud of them both!"

"What about Karen?" asked Larry.

The aunt turned bright red. "How dare you mention her name in my presence."

Evan laughed. "Karen has a masters degree in accounting." Not my area, but I sympathized.

Larry's best friend spoke up. "Listen, Larry. The problem is, it's not cool to do math. Business degrees, they're cool. You know, Internet start-ups and all. Theater degrees, that's cool. You wear black clothes and talk about Pinter. But math? It's not cool. Nothing is cool until everyone is doing it."

Larry wrung his hands.

"You don't understand. I don't have a choice. I am not choosing to do mathematics. Math has chosen me. When I saw that epsilon delta definition of continuity, it was like I had known it all my life. Here is what the professor was really talking about when he drew all these pictures. This is rigorous definition. It felt so good. It's not up to me anymore."

"Look, Larry," I said. "Do you want this to be you?" I showed him the pictures of mathematicians, the addicts with their white pallor from sitting under fluorescent lights for years at a time. Some were barely able to lift their eyes from the books in front of them as the camera clicked away. Their clothes, stained with coffee, made it clear they were unaware that fashion was an evolving concept.

But he was unmoved. "That's exactly what I want to be," he said.

I sighed. "Okay, Larry, I have no choice." I strapped him into the Barcolounger and turned on the TV. I kept him there for two weeks; mostly reruns of "Brady Bunch" and "Welcome Back Kotter." By the time we were done, spittle dripped from the side of his mouth. His brain had been washed clean. Unfortunately, it had been washed so clean that medical school was no longer an option. Larry did go to a successful career with Seven Eleven, primarily mopping up the slushy spills at the Cherry Hill store. And I know that he's happier for it.

But Larry's story is just one among many. These dangers are real. Do you know where your children are? Are you sure they are watching TV, and not sitting in on a seminar, or leafing through a math text?

If we are vigilant, we can prevent mathematics from spreading any further. But we will need to fight the minions of mathematics at every turn. We will need the entertainment industry to continue to hype over intellectual curiosity. We will need to inundate children with the belief that being good at math is something to be ashamed of. We will need to convince that there is nothing wrong with mathematical illiteracy. So far, so good.

III. Graffiti

Edward Burger, The Mathematical Intelligencer, Vol. 20, Number 2, Spring 1998, p. 60.

Graffiti provide a window into the soul of their creator. Given this observation, I was unable to contain my smile upon viewing the above graffiti on the outside wall of a building in Austin, Texas, back in the summer of 1987. On this tenth anniversary, I wish to celebrate the work of this wall artist and its wonderful hidden life lessons.

* *Getting the person-on-the-street to take a look at mathematics.* Certainly the artist's desire to deface property is only exceeded by the artist's passion for mathematics. In the eyes of our wall painter, mathematics is worthy of the attention of the masses. This enthusiasm must have been the impulse for the artist to move beyond scribbling math on napkins to spray painting math on walls. Wouldn't it be great if more mathematicians would actively share their passion for mathematics with the world at large?

* *Shaking well before the final coat.* Our wall artist embarks upon a problem without advance knowledge of how the issue would be resolved. Of course new discoveries are only made after numerous failed attempts. Our painter has a firm grasp on both the spray can and the power of trying without fear of failing. Wouldn't it be great

if more teachers would inspire their students to be brave enough to experiment (and even fail)?

* *Reveling in the shock factor.* WHOOPS! The artist shares with the onlooker the surprise of the realization that the problem at hand was more challenging than first thought. Our wall painter is rather mathematically mature: instead of defacing the wall with a solution known to be wrong, the painter eagerly admits to all that the line of attack did not pan out. Wouldn't it be great if more students would be strong enough to curb the powerful temptation to record an answer they know to be incorrect for the sole purpose of writing something down?

* *Hitting the brick wall.* The wall artist has one of the key ingredients to succeed in mathematics: tenacity. Our painter does not give up or become frustrated when faced with a mathematical impasse. Rather than a typical more colorful epithet, we see the thoughtful proclamation that further insights are required. Wouldn't it be great if more people had a sense that the journey through mathematics is an ongoing one through the uncharted reaches of thought?

* *Painting new pictures.* As all great mathematics should, the artist's work leads us to ponder new and interesting questions. Suppose we define the graffiti curve to be

$$y^4 + 18 - 16 \cdot 16 = 0.$$

Can you show that there are no integer points on this curve? Are there any rational points on the graffiti curve? Wouldn't it be great if each piece of mathematics displayed would inspire one person to ask one new question?

Where is our graffiti artist today? Perhaps the wall painter is now an algebraic geometer at some university or serving time at some other institution. In either case, the painter accomplished something truly spectacular: all who walked by that building, for one brief moment, tried to factor a polynomial.

In the best of all possible worlds, this writing on the wall would have remained to inspire interesting thoughts and conversations among generations of both math fans and math foes. Sadly, this is not the best of all possible worlds. I took this photograph early one Sunday morning. To my surprise, three days later the brick wall was completely painted over with light blue paint to cover the graffiti. That garish blue color covers the entire side of the building to this very day. As people stroll by the wall today, they are unaware of the buried treasure which they pass.

IV. Mathematical Jokes

1. Definitions

Let's start with general definitions.

Mathematics is made of 50 percent formulas, 50 percent proofs, and 50 percent imagination.

An engineer thinks that his equations are an approximation to reality. A physicist thinks reality is an approximation to his equations. A mathematician doesn't care.

Mathematicians are like Frenchmen: whatever you say to them, they translate it into their own language, and forthwith it means something entirely different. (Goethe)

Mathematics is the art of giving the same name to different things. – J. H. Poincare

A topologist is a person who doesn't know the difference between a coffee cup and a doughnut.

A law of conservation of difficulties: there is no easy way to prove a deep result.

A tragedy of mathematics is a beautiful conjecture ruined by an ugly fact.

Philosophy is a game with objectives and no rules.

Mathematics is a game with rules and no objectives.

Math is like love; a simple idea, but it can get complicated.

Mathematics is like checkers in being suitable for the young, not too difficult, amusing, and without peril to the state. (Plato)

Math is the language God used to write the universe.

A mathematician, a physicist, an engineer went again to the races and laid their money down. Commiserating in the bar after the race, the engineer says, "I don't understand why I lost all my money. I measured all the horses and calculated their strength and mechanical advantage and figured out how fast they could run..."

The physicist interrupted him: "...but you didn't take individual variations into account. I did a statistical analysis of their previous performances and bet on the horses with the highest probability of winning..."

"...so if you're so hot why are you broke?" asked the engineer. But before the argument can grow, the mathematician takes out his pipe and they get a glimpse of his well-fattened wallet. Obviously here was a man who knows something about horses. They both demanded to know his secret.

"Well," he says, "first I assumed all the horses were identical and spherical..."

An engineer, a physicist and a mathematician are staying in a hotel.

The engineer wakes up and smells smoke. He goes out into the hallway and sees a fire, so he fills a trash can from his room with water and douses the fire. He goes back to bed.

Later, the physicist wakes up and smells smoke. He opens his door and sees a fire in the hallway. He walks down the hall to a fire hose and after calculating the flame velocity, distance, water pressure, trajectory, etc., extinguishes the fire with the minimum amount of water and energy needed.

Later, the mathematician wakes up and smells smoke. He goes to the hall, sees the fire and then the fire hose. He thinks for a moment and then exclaims, "Ah, a solution exists!" and then goes back to bed.

A physicist and a mathematician are sitting in a faculty lounge. Suddenly, the coffee machine catches on fire. The physicist grabs a bucket and leaps towards the sink, fills the bucket with water and puts out the fire. Second day, the same two sit in the same lounge. Again, the coffee machine catches on fire. This time, the mathematician stands up, gets a bucket, hands the bucket to the physicist, thus reducing the problem to a previously solved one.

A biologist, a physicist and a mathematician were sitting in a street cafe watching the crowd. Across the street they saw a man and a woman entering a building. Ten minutes later they reappeared together with a third person.

"They have multiplied," said the biologist.

"Oh no, an error in measurement," the physicist sighed.

"If exactly one person enters the building now, it will be empty again," the mathematician concluded.

A mathematician is asked to design a table. He first designs a table with no legs. Then he designs a table with infinitely many legs. He spends the rest of his life generalizing the results for the table with N legs (where N is not necessarily a natural number).

A mathematician, a physicist, and an engineer were traveling through Scotland when they saw a black sheep through the window of the train.

"Aha," says the engineer, "I see that Scottish sheep are black."

"Hmm," says the physicist, "You mean that some Scottish sheep are black."

"No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"

A mathematician, scientist, and engineer are each asked: “Suppose we define a horse’s tail to be a leg. How many legs does a horse have?” The mathematician answers “5”; the scientist “1”; and the engineer says, “But you can’t do that!”

A mathematician, a physicist, and an engineer are all given identical rubber balls and told to find the volume. They are given anything they want to measure it, and have all the time they need. The mathematician pulls out a measuring tape and records the circumference. He then divides by two times pi to get the radius, cubes that, multiplies by pi again, and then multiplies by four-thirds and thereby calculates the volume. The physicist gets a bucket of water, places 1.00000 gallons of water in the bucket, drops in the ball, and measures the displacement to six significant figures. And the engineer? He writes down the serial number of the ball, and looks it up.

A Mathematician (M) and an Engineer (E) attend a lecture by a Physicist. The topic concerns Kulza-Klein theories involving physical processes that occur in spaces with dimensions of 9, 12 and even higher. The M is sitting, clearly enjoying the lecture, while the E is frowning and looking generally confused and puzzled. By the end the E has a terrible headache. At the end, the M comments about the wonderful lecture.

E: “How do you understand this stuff?”

M: “I just visualize the process.”

E: “How can you POSSIBLY visualize something that occurs in 9-dimensional space?”

M: “Easy, first visualize it in N-dimensional space, then let N go to 9.”

A team of engineers were required to measure the height of a flag pole. They only had a measuring tape, and were getting quite frustrated trying to keep the tape along the pole. It kept falling down, etc. A mathematician comes along, finds out their problem, and proceeds to remove the pole from the ground and measure it easily. When he leaves, one engineer says to the other: “Just like a mathematician! We need to know the height, and he gives us the length!"

A mathematician and a physicist agree to a psychological experiment. The (hungry) mathematician is put in a chair in a large empty room and his favorite meal, perfectly prepared, is placed at the other end of the room. The psychologist explains, "You are to remain in your chair. Every minute, I will move your chair to a position halfway between its current location and the meal." The mathematician looks at the psychologist in disgust. "What? I'm not going to go through this. You know I'll never reach the food!" And he gets up and storms out. The psychologist ushers the physicist in. He explains the situation, and the physicist's eyes light up and he starts drooling. The psychologist is a bit confused. "Don't you realize that you'll never reach the food?" T he physicist smiles and replies: "Of course! But I'll get close enough for all practical purposes!"

One day a farmer called up an engineer, a physicist, and a mathematician and asked them to fence of the largest possible area with the least amount of fence.

The engineer made the fence in a circle and proclaimed that he had the most efficient design.

The physicist made a long, straight line and proclaimed "We can assume the length is infinite..." and pointed out that fencing off half of the Earth was certainly a more efficient way to do it.

The Mathematician just laughed at them. He built a tiny fence around himself and said "I declare myself to be on the outside."

The physicist and the engineer are in a hot-air balloon. Soon, they find themselves lost in a canyon somewhere. They yell out for help: "Hellllooooooo! Where are we?"

15 minutes later, they hear an echoing voice: "Hellllooooooo! You're in a hot-air balloon!!"

The physicist says, "That must have been a mathematician."

The engineer asks, "Why do you say that?"

The physicist replied: “The answer was absolutely correct, and it was utterly useless.”

Several scientists were asked to prove that all odd integers higher than 2 are prime.

Mathematician: 3 is a prime, 5 is a prime, 7 is a prime, and by induction - every odd integer higher than 2 is a prime.

Physicist: 3 is a prime, 5 is a prime, 7 is a prime, 9 is an experimental error, 11 is a prime. Just to be sure, try several randomly chosen numbers: 17 is a prime, 23 is a prime...

Engineer: 3 is a prime, 5 is a prime, 7 is a prime, 9 is an approximation to a prime, 11 is a prime,...

Programmer (reading the output on the screen): 3 is a prime, 3 is a prime, 3 a is prime, 3 is a prime....

Biologist: 3 is a prime, 5 is a prime, 7 is a prime, 9 – results have not arrived yet,...

Psychologist: 3 is a prime, 5 is a prime, 7 is a prime, 9 is a prime but tries to suppress it,...

Chemist (or Dan Quayle): What’s a prime?

Politician: “Some numbers are prime.. but the goal is to create a kinder, gentler society where all numbers are prime... ”

Programmer: “Wait a minute, I think I have an algorithm from Knuth on finding prime numbers... just a little bit longer, I’ve found the last bug... no, that’s not it... ya know, I think there may be a compiler bug here - oh, did you want IEEE-998.0334 rounding or not? - was that in the spec? - hold on, I’ve almost got it - I was up all night working on this program, ya know... now if management would just get me that new workstation that just came out, I’d be done by now... etc., etc. ...”

(Two is the oddest prime of all, because it’s the only one that’s even!)

Dean, to the physics department. “Why do I always have to give you guys so much money, for laboratories and expensive equipment and stuff. Why couldn’t you be like the math. department - all they need is money for pencils, paper and waste-paper baskets. Or even better, like the philosophy department. All they need are pencils and paper."

New York (CNN). At John F. Kennedy International Airport today, a high school mathematics teacher was arrested trying to board a flight while in possession of a compass, a protractor and a graphical calculator. According to law enforcement officials, he is believed to have ties to the Al-Gebra network. He will be charged with carrying weapons of math instruction. It was later discovered that he taught the students to solve their problem with the help of radicals!

A mathematician organizes a lottery in which the prize is an infinite amount of money. When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: "1 dollar now, 1/2 dollar next week, 1/3 dollar the week after that..."

A Mathematician was put in a room. The room contains a table and three metal spheres about the size of a softball. He was told to do whatever he wants with the balls and the table in one hour. After an hour, the balls are arranges in a triangle at the center of the table. The same test is given to a Physicist. After an hour, the balls are stacked one on top of the other in the center of the table. Finally, an Engineer was tested. After an hour, one of the balls is broken, one is missing, and he's carrying the third out in his lunchbox.

A mathematician decides he wants to learn more about practical problems. He sees a seminar with a nice title: "The Theory of Gears." So he goes. The speaker stands up and begins, "The theory of gears with a real number of teeth is well known ..."

When a statistician passes the airport security check, they discover a bomb in his bag. He explains. "Statistics shows that the probability of a bomb being on an airplane is 1/1000. However, the chance that there are two bombs at one plane is 1/1000000. So, I am much safer..."

What is the difference between a Psychotic, a Neurotic and a mathematician? A Psychotic believes that 2+2=5. A Neurotic knows that 2+2=4, but it kills him. A mathematician simply changes the base.

A mathematician belives nothing until it is proven
A physicist believes everything until it is proven wrong
A chemist doesn't care
A biologist doesn't understand the question.

To mathematicians, solutions mean finding the answers. But to chemists, solutions are things that are still all mixed up.

3. Mathematical education

These sketches demonstrate how desperately we want to push the math into the public education, and the struggle and passion of math. students.

The Evolution of Math Teaching

- 1960s: A peasant sells a bag of potatoes for $10. His costs amount to 4/5 of his selling price. What is his profit?
- 1970s: A farmer sells a bag of potatoes for $10. His costs amount to 4/5 of his selling price, that is, $8. What is his profit?
- 1970s (new math): A farmer exchanges a set P of potatoes with set M of money. The cardinality of the set M is equal to 10, and each element of M is worth $1. Draw ten big dots representing the elements of M. The set C of production costs is composed of two big dots less than the set M. Represent C as a subset of M and give the answer to the question: What is the cardinality of the set of profits?
- 1980s: A farmer sells a bag of potatoes for $10. His production costs are $8, and his profit is $2. Underline the word "potatoes" and discuss with your classmates.

- 1990s: A farmer sells a bag of potatoes for \$10. His or her production costs are 0.80 of his or her revenue. On your calculator, graph revenue vs. costs. Run the POTATO program to determine the profit. Discuss the result with students in your group. Write a brief essay that analyzes this example in the real world of economics.

(Anon: adapted from The American Mathematical Monthly, Vol. 101, No. 5, May 1994 (Reprinted by STan Kelly-Bootle in Unix Review, Oct 94)

Top excuses for not doing homework:

- I accidentally divided by zero and my paper burst into flames.
- Isaac Newton's birthday.
- I could only get arbitrarily close to my textbook. I couldn't actually reach it.
- I have the proof, but there isn't room to write it in this margin.
- I was watching the World Series and got tied up trying to prove that it converged.
- I have a solar powered calculator and it was cloudy.
- I locked the paper in my trunk but a four-dimensional dog got in and ate it.
- I could have sworn I put the homework inside a Klein bottle, but this morning I couldn't find it.

Warning! It is against the rule to use these excuses in my classes! A. Ch.

A professor's enthusiasm for teaching precalculus varies inversely with the likelihood of his having to do it.

A student comes to the department with a shiny new cup, the sort of which you get when having won something. He explained: I won it in the MD Math Contest. They asked what 7 + 7 is. I said 12 and got 3rd place!

Two male mathematicians are in a bar. The first one says to the second that the average person knows very little about basic mathematics. The second one disagrees, and claims that most people can cope with a reasonable amount of math.

The first mathematician goes off to the washroom, and in his absence the second calls over the waitress. He tells her that in a few minutes, after his friend has returned, he will call her over and ask her a question. All she has to do is answer one third x cubed.

She repeats "one thir – dex cue"?

He repeats "one third x cubed".

Her: 'one thir dex cuebd'? Yes, that's right, he says. So she agrees, and goes off mumbling to herself, "one thir dex cuebd...".

The first guy returns and the second proposes a bet to prove his point, that most people do know something about basic math. He says he will ask the blonde waitress an integral, and the first laughingly agrees. The second man calls over the waitress and asks "what is the integral of x squared?"

The waitress says "one third x cubed" and while walking away, turns back and says over her shoulder "plus a constant!"

A somewhat advanced society has figured how to package basic knowledge in pill form.

A student, needing some learning, goes to the pharmacy and asks what kind of knowledge pills are available. The pharmacist says "Here's a pill for English literature." The student takes the pill and swallows it and has new knowledge about English literature!

"What else do you have?" asks the student.

"Well, I have pills for art history, biology, and world history," replies the pharmacist.

The student asks for these, and swallows them and has new knowledge about those subjects.

Then the student asks, "Do you have a pill for math?"

The pharmacist says "Wait just a moment", and goes back into the storeroom and brings back a whopper of a pill and plunks it on the counter.

"I have to take that huge pill for math?" inquires the student.

The pharmacist replied "Well, you know math always was a little hard to swallow."

Golden rule for math teachers: You must tell the truth, and nothing but the truth, but not the whole truth.

A math professor is one who talks in someone else's sleep.

Q: What do you get when you add 2 apples to 3 apples?
A: Answer: A senior high school math problem.

Quotes from math students and lecturers

"This is a one line proof...if we start sufficiently far to the left."

"The problems for the exam will be similar to the discussed in the class. Of course, the numbers will be different. But not all of them. Pi will still be 3.14159... "

4. Seminar semantics, etc.

A lecturer: "Now we'll prove the theorem. In fact I'll prove it all by myself."

How to prove it. Guide for lecturers.

Proof by vigorous handwaving:
Works well in a classroom or seminar setting.

Proof by example:
The author gives only the case n = 2 and suggests that it contains most of the ideas of the general proof.

Proof by omission:
"The reader may easily supply the details" or "The other 253 cases are analogous"

Proof by intimidation:
"Trivial."

Proof by cumbersome notation:
Best done with access to at least four alphabets and special symbols.

Proof by exhaustion:
An issue or two of a journal devoted to your proof is useful.

Proof by reference to inaccessible literature:
The author cites a simple corollary of a theorem to be found in a privately circulated memoir of the Slovenian Philological Society, 1883.

Proof by accumulated evidence:
Long and diligent search has not revealed a counterexample.

Proof by mutual reference:
In reference A, Theorem 5 is said to follow from Theorem 3 in reference B, which is shown to follow from Corollary 6.2 in reference C, which is an easy consequence of Theorem 5 in reference A.

Proof by metaproof:
A method is given to construct the desired proof. The correctness of the method is proved by any of these techniques.

Proof by ghost reference:
Nothing even remotely resembling the cited theorem appears in the reference given.

Proof by semantic shift:
Some of the standard but inconvenient definitions are changed for the statement of the result.

Proof by appeal to intuition:
Cloud-shaped drawings frequently help here.

Dictionary of Definitions of Terms Commonly Used in Math Lectures

The following is a guide to terms which are commonly used but rarely defined. In the search for proper definitions for these terms we found no authoritative, nor even recognized, source. Thus, we followed the advice of mathematicians handed down from time immortal: "Wing It."

CLEARLY:

I don't want to write down all the "in- between" steps.

TRIVIAL:

If I have to show you how to do this, you're in the wrong class.

OBVIOUSLY:

I hope you weren't sleeping when we discussed this earlier, because I refuse to repeat it.

RECALL:

I shouldn't have to tell you this, but for those of you who erase your memory tapes after every test...

WLOG (Without Loss Of Generality):

I'm not about to do all the possible cases, so I'll do one and let you figure out the rest.

IT CAN EASILY BE SHOWN:

Even you, in your finite wisdom, should be able to prove this without me holding your hand.

CHECK or CHECK FOR YOURSELF:

This is the boring part of the proof, so you can do it on your own time.

SKETCH OF A PROOF:

I couldn't verify all the details, so I'll break it down into the parts I couldn't prove.

HINT:

The hardest of several possible ways to do a proof.

BRUTE FORCE (AND IGNORANCE):

Four special cases, three counting arguments, two long inductions, "and a partridge in a pair tree."

SOFT PROOF:

One third less filling (of the page) than your regular proof, but it requires two extra years of course work just to understand the terms.

ELEGANT PROOF:

Requires no previous knowledge of the subject matter and is less than ten lines long.

SIMILARLY:

At least one line of the proof of this case is the same as before.

CANONICAL FORM:

4 out of 5 mathematicians surveyed recommended this as the final form for their students who choose to finish.

TFAE (The Following Are Equivalent):

If I say this it means that, and if I say that it means the other thing, and if I say the other thing...

BY A PREVIOUS THEOREM:

I don't remember how it goes (come to think of it I'm not really sure we did this at all), but if I stated it right (or at all), then the rest of this follows.

TWO LINE PROOF:

I'll leave out everything but the conclusion, you can't question 'em if you can't see 'em.

BRIEFLY:

I'm running out of time, so I'll just write and talk faster.

LET'S TALK THROUGH IT:

I don't want to write it on the board lest I make a mistake.

PROCEED FORMALLY:

Manipulate symbols by the rules without any hint of their true meaning (popular in pure math courses).

QUANTIFY:

I can't find anything wrong with your proof except that it won't work if x is a moon of Jupiter (Popular in applied math courses).

PROOF OMITTED:

Trust me, It's true.

Traditional - contemporary math dictionary.

WHAT'S OUT AND WHAT'S IN FOR MATHEMATICAL TERMS

by Michael Stueben (November 7, 1994)

Today it is considered an egregious faux pas to speak or write in the crude antedated terms of our grandfathers. To assist the isolated student and the less sophisticated teacher, I have prepared the following list of currently fashionable mathematical terms in academia. I pass this list on to the general public as a matter of charity and in the hope that it will lead to more refined elucidation from young scholars.

thinking: hypothesizing.
proof by contradiction or indirect proof: reductio ad absurdum.
mistake: non sequitur.
starting place: handle.
with corresponding changes: mutatis mutandis.
counterexample: pathological exception.
consequently: ipso facto.
swallowing results: digesting proofs.
therefore: ergo.
has an easy-to-understand, but hard-to-find solution: obvious.
has two easy-to-understand, but hard-to-find solutions: trivial.
truth: tautology.
empty: vacuous.
drill problems: plug-and-chug work.
criteria: rubric.
example: substantive instantiation. similar structure: homomorphic.
very similar structure: isomorphic.
same area: isometric.
arithmetic: number theory.
count: enumerate.
one: unity.
generally/specifically: globally/locally.
constant: invariant.
bonus result: corollary.
distance: metric measure.

several: a plurality.

function/argument: operator/operand.

separation/joining: bifurcation/confluence.

fourth power or quartic: biquadratic.

random: stochastic.

unique condition: a singularity.

uniqueness: unicity.

tends to zero: vanishes.

tip-top point: apex.

half-closed: half-open.

concave: non-convex.

rectangular prisms: parallelepipeds.

perpendicular (adj.): orthogonal.

perpendicular (n.): normal.

Euclid: Descartes.

Fermat: Wiles.

path: trajectory.

shift: rectilinear translation.

similar: homologous.

very similar: congruent.

whopper-jawed: skew or oblique.

change direction: perturb.

join: concatenate.

approximate to two or more places: accurate.

high school geometry or plane geometry: geometry of the Euclidean plane under the Pythagorean metric. clever scheme: algorithm.

initialize to zero: zeroize.

decimal: denary.

alphabetical order: lexical order.

a divide-and-conquer method: an algorithm of logarithmic order.

student ID numbers: witty passwords.

numerology and number sophistry: descriptive statistics

Special thanks to Peter Braxton who got me started writing this stuff and who contributed five of the items above.

Professional secrets

The highest moments in the life of a mathematician are the first few moments after one has proved the result, but before one finds the mistake.

Golden rule of deriving: never trust any result that was proved after 11 PM.

The professional quality of a mathematician is inversely proportional to the importance it attaches to space and equipment.

Relations between pure and applied mathematicians are based on trust and understanding. Namely, pure mathematicians do not trust applied mathematicians, and applied mathematicians do not understand pure mathematicians.

Some mathematicians become so tense these days that they that they do not go to sleep during seminars.

If I have seen farther than others, it is because I was standing on the shoulders of giants.
– Isaac Newton

In the sciences, we are now uniquely privileged to sit side by side with the giants on whose shoulders we stand.
– Gerald Holton

If I have not seen as far as others, it is because giants were standing on my shoulders.
– Hal Abelson

Mathematicians stand on each other's shoulders.
– Gauss

Mathematicians stand on each other's shoulders while computer scientists stand on each other's toes.
– Richard Hamming

It has been said that physicists stand on one another's shoulders. If this is the case, then programmers stand on one another's toes, and software engineers dig each other's graves.

– Unknown

These days, even the most pure and abstract mathematics is in danger to be applied.

The reason that every major university maintains a department of mathematics is that it is cheaper to do this than to institutionalize all those people.

5. Theorems

Here, the powerful mathematical methods are successively applied to the "real life problems".

Interesting Theorem:

All positive integers are interesting.

Proof:

Assume the contrary. Then there is a lowest non-interesting positive integer. But, hey, that's pretty interesting! A contradiction.

Boring Theorem:

All positive integers are boring.

Proof:

Assume the contrary. Then there is a lowest non-boring positive integer. Who cares!

Discovery:

Mathematicians have announced the existence of a new whole number which lies between 27 and 28. "We don't know why it's there or what it does," says Cambridge mathematician, Dr. Hilliard Haliard, "we only know that it doesn't behave properly when put into equations, and that it is divisible by six, though only once."

Theorem:

There are two groups of people in the world; those who believe that the world can be divided into two groups of people, and those who don't.

Theorem:

The world is divided into two classes: people who say "The world is divided into two classes", and people who say: The world is divided into two classes: people who say: "The world is divided into two classes", and people who say: The world is divided into two classes: people who say ...

There are three kinds of people in the world; those who can count and those who can't.

There are 10 kinds of people in the world, those who understand binary math, and those who don't.

There really are only two types of people in the world, those that DON'T DO MATH, and those that take care of them.

Cat Theorem:

A cat has nine tails.

Proof:

No cat has eight tails. A cat has one tail more than no cat. Therefore, a cat has nine tails.

Salary Theorem

The less you know, the more you make.

Proof:

Postulate 1: Knowledge is Power.

Postulate 2: Time is Money.As every engineer knows: Power = Work / Time

And since Knowledge = Power and Time = Money

It is therefore true that Knowledge = Work / Money .

Solving for Money, we get:

Money = Work / Knowledge

Thus, as Knowledge approaches zero, Money approaches infinity, regardless of the amount of Work done.

Q: How do you tell that you are in the hands of the Mathematical Mafia?

A: They make you an offer that you can't understand.

Notes on the horse colors problem

Lemma 1. All horses are the same color. (Proof by induction)

Proof. It is obvious that one horse is the same color. Let us assume the proposition P(k) that k horses are the same color and use this to imply that k+1 horses are the same color. Given the set of k+1 horses, we remove one horse; then the remaining k horses are the same color, by hypothesis. We remove another horse and replace the first; the k horses, by hypothesis, are again the same color. We repeat this until by exhaustion the k+1 sets of k horses have been shown to be the same color. It follows that since every horse is the same color as every other horse, P(k) entails P(k+1). But since we have shown P(1) to be true, P is true for all succeeding values of k, that is, all horses are the same color.

Theorem 1. Every horse has an infinite number of legs. (Proof by intimidation.)

Proof. Horses have an even number of legs. Behind they have two legs and in front they have fore legs. This makes six legs, which is certainly an odd number of legs for a horse. But the only number that is both odd and even is infinity. Therefore horses have an infinite number of legs. Now to show that this is general, suppose that somewhere there is a horse with a finite number of legs. But that is a horse of another color, and by the lemma that does not exist.

Corollary 1. Everything is the same color.

Proof. The proof of lemma 1 does not depend at all on the nature of the object under consideration. The predicate of the antecedent of the universally-quantified conditional 'For all x, if x is a horse, then x is the same color,' namely 'is a horse' may be generalized to 'is anything' without affecting the validity of the proof; hence, 'for all x, if x is anything, x is the same color.'

Corollary 2. Everything is white.

Proof. If a sentential formula in x is logically true, then any particular substitution instance of it is a true sentence. In particular then: 'for all x, if x is an elephant, then x is the same color' is true. Now it is manifestly axiomatic that white elephants exist (for proof by blatant assertion consult Mark Twain 'The Stolen White Elephant'). Therefore all elephants are white. By corollary 1 everything is white.

Theorem 2. Alexander the Great did not exist and he had an infinite number of limbs.

Proof. We prove this theorem in two parts. First we note the obvious fact that historians always tell the truth (for historians always take a stand, and therefore they cannot lie). Hence we have the historically true sentence, 'If Alexander the Great existed, then he rode a black horse Bucephalus.' But we know by corollary 2 everything is white; hence Alexander could not have ridden a black horse. Since the consequent of the conditional is false, in order for the whole statement to be true the antecedent must be false. Hence Alexander the Great did not exist.

We have also the historically true statement that Alexander was warned by an oracle that he would meet death if he crossed a certain river. He had two legs; and 'forewarned is four-armed.' This gives him six limbs, an even number, which is certainly an odd number of limbs for a man. Now the only number which is even and odd is infinity; hence Alexander had an infinite number of limbs. We have thus proved that Alexander the Great did not exist and that he had an infinite number of limbs.

According to statistics, there are 42 million alligator eggs laid every year. Of those, only about half get hatched. Of those that hatch, three fourths of them get eaten by predators in the first 36 days. And of the rest, only 5 percent get to be a year old for one reason or another. Isn't statistics wonderful? If it weren't for statistics, we'd be eaten by alligators!

6. Playground

Let's play with math objects!

An insane mathematician gets on a bus and starts threatening everybody: "I'll integrate you! I'll differentiate you!!!" Everybody gets scared and runs away. Only one lady stays. The guy comes up to her and says: "Aren't you scared, I'll integrate you, I'll differentiate you!!!" The lady calmly answers: "No, I am not scared, I am e^x ."

More advanced and more New York style story:

A constant function and e^x are walking on Broadway. Then suddenly the constant function sees a differential operator approaching and runs away. So e^x follows him and asks why the hurry. "Well, you see, there's this differential operator coming this way, and when we meet, he'll differentiate me and nothing will be left of me...!" "Ah," says e^x, "he won't bother ME, I'm e to the x!" and he walks on. Of course he meets the differential operator after a short distance.

e^x: "Hi, I'm e^x"

diff.op.: "Hi, I'm $\frac{d}{dy}$"

"The number you have dialed is imaginary. Please rotate your phone 90 degrees and try again."

The shortest math joke: let epsilon be < 0

Funny formulas

The limit as 3 goes to 4 of 3^2 is 16.

(For native LaTex speakers: $\lim_{3\to 4} 3^2 = 16$)

$1 + 1 = 3$, for sufficiently large one's.

The combination of the Einstein and Pythagoras discoveries:

$E = mc^2 = m(a^2 + b^2)$

2 and 2 is 22

The limit as n goes to infinity of $\sin(x)/n$ is 6.

Proof: cancel the n in the numerator and denominator.

As x goes to zero, the limit of $8/x$ is ∞ (infinity), then the limit (as x goes to zero) of Z/x is N

Q: How many times can you subtract 7 from 83, and what is left afterwards?

A: I can subtract it as many times as I want, and it leaves 76 every time.

A Neanderthal child rode to school with a boy from Hamilton. When his mother found out she said, "What did I tell you? If you commute with a Hamiltonian you'll never evolve!"

In modern mathematics, algebra has become so important that numbers will soon only have symbolic meaning.

A circle is a round straight line with a hole in the middle.

In the topologic hell the beer is packed in Klein's bottles.

Q: Why did the chicken cross the Moebius strip?

A: To get to the other ... er, um ...

Two mathematicians are studying a convergent series. The first one says: "Do you realize that the series converges even when all the terms are made positive?" The second one asks: "Are you sure?" "Absolutely!"

Q: What does the zero say to the the eight?

A: Nice belt!

Life is complex: it has both real and imaginary components.

Math problems? Call 1-800-[(10x)(13i)2]-[sin(xy)/2.362x].

"Divide fourteen sugar cubes into three cups of coffee so that each cup has an odd number of sugar cubes in it." "That's easy: one, one, and twelve." "But twelve isn't odd!" "Twelve is an odd number of cubes to put in a cup of coffee..."

A statistician can have his head in an oven and his feet in ice, and he will say that on the average he feels fine.

Q: Did you hear the one about the statistician?

A: Probably....

The light bulb problem

Q: How many mathematicians does it take to screw in a light bulb?

A1: None. It's left to the reader as an exercise.

A2: None. A mathematician can't screw in a light bulb, but he can easily prove the work can be done.

A3: One. He gives it to four programmers, thereby reducing the problem to the already solved (ask a programmer, how)

A4: The answer is intuitively obvious

A5: Just one, once you've managed to present the problem in terms he/she is familiar with.

A6: In earlier work, Wiener [1] has shown that one mathematician can change a light bulb.

If k mathematicians can change a light bulb, and if one more simply watches them do it, then $k + 1$ mathematicians will have changed the light bulb.

Therefore, by induction, for all n in the positive integers, n mathematicians can change a light bulb.

Bibliography:

[1] Weiner, Matthew P,...

How many numerical analysts does it take to replace a light bulb?

3.9967: (after six iterations).

How many classical geometers does it take to replace a light bulb?

None: You can't do it with a straight edge and a compass.

How many constructivist mathematicians does it take to replace a light bulb?

None: They do not believe in infinitesimal rotations.

How many topologists does it take to screw in a light bulb?
Just one. But what will you do with the doughnut?

How many analysts does it take to screw in a light bulb?
Three: One to prove existence, one to prove uniqueness and one to derive a nonconstructive algorithm to do it.

How many Bourbakists does it take to replace a light bulb?
Changing a light bulb is a special case of a more general theorem concerning the maintain and repair of an electrical system. To establish upper and lower bounds for the number of personnel required, we must determine whether the sufficient conditions of Lemma 2.1 (Availability of personnel) and those of Corollary 2.3.55 (Motivation of personnel) apply. Iff these conditions are met, we derive the result by an application of the theorems in Section 3.1123. The resulting upper bound is, of course, a result in an abstract measure space, in the weak-* topology.

How many professors does it take to replace a light bulb?
One: With eight research students, two programmers, three post-docs and a secretary to help him.

How many university lecturers does it take to replace a light bulb?
Four: One to do it and three to co-author the paper.

How many graduate students does it take to replace a light bulb?
Only one: But it takes nine years.

How many math department administrators does it take to replace a light bulb?
None: What was wrong with the old one then?

How we do it ...
Aerodynamicists do it in drag.
Algebraists do it by symbolic manipulation.

Algebraists do it in a ring, in fields, in groups.

Analysts do it continuously and smoothly.

Applied mathematicians do it by computer simulation.

Banach spacers do it completely.

Bayesians do it with improper priors.

Catastrophe theorists do it falling off part of a sheet.

Combinatorists do it as many ways as they can.

Complex analysts do it between the sheets

Computer scientists do it depth-first.

Cosmologists do it in the first three minutes.

Decision theorists do it optimally.

Functional analysts do it with compact support.

Galois theorists do it in a field.

Game theorists do it by dominance or saddle points.

Geometers do it with involutions.

Geometers do it symmetrically.

Graph theorists do it in four colors.

Hilbert spacers do it orthogonally.

Large cardinals do it inaccessibly.

Linear programmers do it with nearest neighbors.

Logicians do it by choice, consistently and completely.

Logicians do it incompletely or inconsistently.

(Logicians do it) or [not (logicians do it)].

Number theorists do it perfectly and rationally.

Mathematical physicists understand the theory of how to do it, but have difficulty obtaining practical results.

Pure mathematicians do it rigorously.

Quantum physicists can either know how fast they do it, or where they do it, but not both.

Real analysts do it almost everywhere

Ring theorists do it non-commutatively.

Set theorists do it with cardinals.

Statisticians probably do it.

Topologists do it openly, in multiply connected domains

Variationists do it locally and globally.

Cantor did it diagonally.
Fermat tried to do it in the margin, but couldn't fit it in.
Galois did it the night before.
Möbius always does it on the same side.
Markov does it in chains.
Newton did it standing on the shoulders of giants.
Turing did it but couldn't decide if he'd finished.

A SLICE OF PI
****
3.14159265358979
1640628620899
23172535940
881097566
5432664
09171
036
5

7. Puns

Q: What's the contour integral around Western Europe?
A: Zero, because all the Poles are in Eastern Europe!
Addendum: Actually, there ARE some Poles in Western Europe, but they are removable!

Q:What is a dilemma?
A: A lemma that proves two results.

Q: What's nonorientable and lives in the sea?
A: Moebius Dick.

Q: What's yellow and equivalent to the Axiom of Choice.
A: Zorn's Lemon.

Q: What's purple and commutes?
A: An abelian grape.

Q: What's yellow, linear, normed and complete?
A: A Bananach space.

Q: What's a polar bear?
A: A rectangular bear after a coordinate transform.

Was General Calculus a Roman war hero?

"What's your favorite thing about mathematics?" "Knot theory." "Yeah, me neither."

Q: Why didn't Newton discover group theory?
A: Because he wasn't Abel.

In Alaska, where it gets very cold, pi is only 3.00. As you know, everything shrinks in the cold. They call it Eskimo pi.

How do you prove in three steps that a sheet of paper is a lazy dog?
1. A sheet of paper is an ink-lined plane.
2. An inclined plane is a slope up.
3. A slow pup is a lazy dog.

A geometer went to the beach to catch the rays and became a TanGent.

8. Anecdotes

Next several stories are attributed to real mathematicians. For most of them, it was impossible to check the truthfulness of the story. Therefore the names are often removed.

In 1915, Emma Noether arrived in Göttingen but was denied the private-docent status. The argument was that a woman cannot attend the University senate (the faculty meetings). Hilbert's reaction was: "Gentlemen! There is nothing wrong to have a woman in the senate. Senate is not a bath."

The following problem can be solved either the easy way or the hard way.

Two trains 200 miles apart are moving toward each other; each one is going at a speed of 50 miles per hour. A fly starting on the front of one of them flies back and forth between them at a rate of 75 miles per hour. It does this until the trains collide and crush the fly to death. What is the total distance the fly has flown?

The fly actually hits each train an infinite number of times before it gets crushed, and one could solve the problem the hard way with pencil and paper by summing an infinite series of distances. The easy way is as follows: Since the trains are 200 miles apart and each train is going 50 miles an hour, it takes 2 hours for the trains to collide. Therefore the fly was flying for two hours. Since the fly was flying at a rate of 75 miles per hour, the fly must have flown 150 miles. That's all there is to it.

When this problem was posed to John von Neumann, he immediately replied, "150 miles." "It is very strange," said the poser, "but nearly everyone tries to sum the infinite series." "What do you mean, strange?" asked Von Neumann. "That's how I did it!"

Another von Neumann quote : Young man, in mathematics you don't understand things, you just get used to them.

The mathematician S. had to move to a new place. His wife didn't trust him very much, so when they stood down on the street with all their things, she asked him to watch their ten trunks, while she get a taxi. Some minutes later she returned. Said the husband:

"I thought you said there were ten trunks, but I've only counted to nine."

The wife said: "No, they're TEN!"

"But I have counted them: 0, 1, 2, ..."

N. had the habit of simply writing answers to homework assignments on the board (the method of solution being, of course, obvious) when he was asked how to solve problems. One time one of his students tried to get more helpful information by asking if there was another way to solve the problem. N. looked blank for a moment, thought, and then answered, "Yes".

In his lecture, ** formulated a theorem and said: "The proof is obvious". Then he thought for a minute, left the lecture room, returned after 15 minutes and happily concluded: "Indeed, it is obvious!"

A famous mathematician was to give a keynote speech at a conference. Asked for an advance summary, he said he would present a proof of Fermat's Last Theorem – but they should keep it under their hats. When he arrived, though, he spoke on a much more prosaic topic. Afterwards the conference organizers asked why he said he'd talk about the theorem and then didn't. He replied this was his standard practice, just in case he was killed on the way to the conference.

A mathematician about his late colleague: "He made a lot of mistakes, but he made them in a good direction. I tried to copy this, but I found out that it is very difficult to make good mistakes."

This story is attributed to Professor Lev Loytiansky, the stage is in Soviet Union in the thirties or forties.

L. organized the seminar in hydrodynamics in his University. Among the regular attendees there were two men in the uniform, obviously military engineers. They never discussed the problems they were working on. But one day they ask L. to help with a math. problem. They explained that the solution of a certain equation oscillated and asked how they should change the coefficients to make it monotonic. L. looked on the equation and said: "Make the wings longer!"

Students asked ** to exclude a part of the course from the final exam. ** agreed. Encouraged by the easy success, the students asked to skip another part of the course, and ** agreed again, and then again. However, in the end of the term he did include all this material in the exam. The class loudly complained: "Dr **, you promised us to skip this stuff!" ** answered: "Yes, I did. But I lied!"

Ernst Eduard Kummer (1810-1893), a German algebraist, was sometimes slow at calculations.. Whenever he had occasion to do simple arithmetic in class, he would get his students to help him. Once he had to find 7 x 9. "Seven times nine," he began, "Seven times nine is er – ah — ah – seven times nine is. . . ." "Sixty-one," a student suggested. Kummer wrote 61 on the board. "Sir," said another student, "it should be sixty-nine." "Come, come, gentlemen, it can't be both," Kummer exclaimed. "It must be one or the other."

This anecdote is attributed to Landau (the Russian physicist Lev not the Göttingen mathematician Edmund).

Landau's group was discussing a bright new theory, and one of junior colleagues of Landau bragged that he had independently discovered the theory a couple of years ago, but did not bother to publish his finding.

"I would not repeat this claim if I were you," Landau replied: "There is nothing wrong if one has not found a solution to a particular problem. However, if one has found it but does not publish it, he shows a poor judgment and inability to understand what important is in modern physics".

9. Limericks.

Limericks are always limericks. (The obscene math limericks were mercilessly excluded.)

A mathematician confided
That the Möbius band is one-sided
And you'll get quite a laugh
If you cut one in half
'Cause it stays in one piece when divided.

'Tis a favorite project of mine
A new value of pi to assign.

I would fix it at 3
For it's simpler, you see,
Than 3 point 1 4 1 5 9

A challenge for many long ages
Had baffled the savants and sages.
Yet at last came the light:
Seems old Fermat was right–
To the margin add 200 pages.

Integral z-squared dz
from 1 to the cube root of 3
times the cosine
of three pi over 9
equals log of the cube root of 'e'.
And it's correct, too.

This poem was written by John Saxon (an author of math textbooks).

$((12 + 144 + 20 + (3 * 4^{(1/2)}))/7) + (5 * 11) = 9^2 + 0$

A Dozen, a Gross and a Score,
plus three times the square root of four,
divided by seven,
plus five times eleven,
equals nine squared and not a bit more.

In arctic and tropical climes,
the integers, addition, and times,
taken (mod p) will yield
a full finite field,
as p ranges over the primes.

Chebychev said it and I'll say it again:
There's always a prime between n and 2n!

A conjecture both deep and profound
Is whether the circle is round;

In a paper by Erdös,
written in Kurdish,
A counterexample is found.
(Note: Erdös is pronounced “Air - dish”)

There once was a number named pi
Who frequently liked to get high.
All he did every day
Was sit in his room and play
With his imaginary friend named i.

Составители: Л. Н. Выгонская, М. Ф. Гольберг,
Л. С. Карпова, А. А. Савченко.

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