Special Functions: An Introduction to the Classical Functions of Mathematical Physics

by Nico Temme
reviewed by Roderick Wong
abridged for academic purposes

About 50 years ago, special functions were considered important in the field of analysis. They occupy half of the classic books such as Whittaker and Watson and Copson. They were, and still are, frequently used in physics and engineering. However, in recent years, many mathematicians, both pure and applied, have held the view that with the invention of powerful computers, special functions are no longer needed or have become obsolete. If this is the prevalent view, then does it mean that we can avoid using the exponential function, the logarithmic function, and the trigonometric functions? Higher transcendental functions such as the gamma function, the Bessel functions, and the Legendre functions are only a level higher than the elementary functions just mentioned. To scientists, these functions play the same role as the elementary functions do to high school students.

Although there are already many books on special functions and some of them have become classics, most of these books are not suitable as texts, and most of the textbooks being used are not written by specialists. The present book was written by Nico Temme, a specialist who has spent 30 years working on this subject. He clearly knows what to include and how to present the material. Properties of special functions should be simple to state, and the formulas should not be too long, although their proofs can be lengthy and complicated. Functions involving too many variables are best excluded from a textbook such as this.

Temme starts Chapter 1 on Bernoulli, Euler, and Stirling numbers. The Bernoulli numbers are discussed in most books on special functions, but Euler and Stirling numbers are rarely mentioned in textbooks on this subject. A particularly attractive inclusion in this chapter is the Boole summation formula. While the Euler summation formula deals with sums of positive terms, the Boole summation formula treats sums of terms with alternating signs.

In Chapter 2, Temme lists some useful theorems in analysis. These theorems are mainly concerned with the interchange of the order of integration or the order of summation and integration. Also included in this chapter is a brief description of a powerful method in asymptotics, the saddle-point method.

Chapter 3 deals with the gamma function. The material in this chapter is standard and can be found in most books on special functions. The only formula here which is usually not discussed in introductory books on special functions is an asymptotic expansion for the ratio of two gamma functions.

Chapter 4 is on differential equations. It begins with the method of separation of variables, which reduces partial differential equations to ordinary differential equations. It then proceeds to discuss the power series solutions to ordinary differential equations near regular and singular points. It ends with short sections on Sturm's comparison theorem, integral representation for solutions of differential equations, and the Liouville transformation. It was a wise decision to include these topics.

Hypergeometric functions are discussed in Chapter 5, including a short introduction to the q-functions, which is currently a hot topic.

Orthogonal polynomials are dealt with in Chapter 6. Except for Legendre polynomials, the classical orthogonal polynomials are mentioned only briefly. It would have been nice to include some of the more recent results on these classical orthogonal polynomials. For instance, the inequality given on page 158,

$$\sqrt{1-x^2}|P_n(x)| \leq \sqrt{\frac{\pi}{2n}},$$

valid for $-1 < x < 1$, $n = 1, 2, \ldots$, has been improved by Antonov and Holševnikov, who showed that the n on the right-hand side can be replaced by $n+1$. Furthermore, Lorch has generalized this result by showing that the ultraspherical polynomial $P_n^{(\lambda)}(x)$ satisfies

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}\{\Gamma(\lambda)\}^{-1}(n+\lambda)^{\lambda-1}$$

for $0 < \lambda < 1$ and $-1 \leq x \leq 1$. Up to now, the most general result in this direction is probably that of Chow, Gatteschi, and Wong. They proved that for $-\frac{1}{2} \leq \alpha,\ \beta \leq 12$ and $-1 \leq x \leq 1$, the Jacobi polynomial satisfies the inequality

$$(\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}} \times |P-n^{(\alpha,\beta)}(\cos\theta)| \leq \frac{\Gamma(q+1)}{\Gamma(\frac{1}{2})} \times \binom{n+q}{n} N^{-q-\frac{1}{2}},$$

where $N = n + \frac{1}{2}(\alpha+\beta+1)$ and $q = \max(\alpha,\beta)$. These results are not too long or complicated to state, and they could have been mentioned (without proofs) at the end of the chapter for interested readers.

In Chapter 7, Temme discusses confluent hypergeometric functions. Instead of the older notation Φ and Ψ used by Whittaker and Watson and by Erdélyi, Temme prefers the M and U functions introduced by J.C.P. Miller and recommended by F.W.J. Olver. As special cases of the confluent hypergeometric functions, he briefly mentions the Whittaker functions, the parabolic cylinder functions, error functions, exponential integrals, Fresnel integrals, and incomplete gamma functions.

Chapter 8 contains the usual material on Legendre functions. The presentation is both clear and to the point.

Among all special functions, Bessel functions are the ones that are probably most frequently used. They are presented in Chapter 9. Even for such a well-studied function, new and interesting results are continually being discovered. For instance, it has recently been proved that the k-th positive zero $j_{\nu,k}$ of the Bessel function $J_\nu(x)$ has the upper and lower bounds

$$\nu - \frac{a_k}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} < j_{\nu,k} < \nu - \frac{a_k}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} + \frac{3}{20}a_k^2\frac{2^{\frac{1}{3}}}{\nu^{\frac{1}{3}}}$$

for all $\nu > 0$ and all $k = 1, 2, \ldots$ where a_k is the k-th negative zero of the Airy function. The surprising fact is that these are exactly the first few terms in the asymptotic expansion

$$j_{\nu,k} \sim \nu - \frac{a_k}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} + \frac{3}{20}a_k^2\frac{2^{\frac{1}{3}}}{\nu^{\frac{1}{3}}} + \ldots$$

as $\nu \to \infty$, k being fixed.

Chapters 10 and 11 are the most unique features of this book. Chapter 10 summarizes various transformations that take the Cartesian coordinates to other coordinate systems such as cylindrical coordinates, spherical coordinates, elliptic cylinder coordinates, parabolic cylinder coordinates, and oblate spheroidal coordinates. In addition, Temme gives formulas for operators such as the gradient, Laplacian, divergence, and curl in different coordinates. This is indeed a very useful chapter for people working in applied fields.

In Chapter 11, Temme considers some special functions that are related to statistical distribution functions. These include the error functions, incomplete gamma functions, and incomplete beta functions. They are related to the normal (Gaussian) distribution, the χ^2-distribution, the beta distribution, and the non-central χ^2-distribution. For each of these special functions, uniform asymptotic expansions are provided. Most of this work was done by Temme himself.

Chapter 12 concerns elliptic integrals and elliptic functions. The presentation here is very attractive: it includes a derivation of the period of a simple pendulum, a discussion of the connection between elliptic integrals, and the iteration by the arithmetic geometric mean (AGM). There is also some information on numerical computation of elliptic integrals.

The final chapter is on numerical aspects of special functions. It begins with a short summary of the literature on software programs for special functions, and lists some packages that are available for symbolic computations. The method for numerical computation based on recurrence relations is discussed in some detail, including a section on Miller's algorithm.

Each chapter includes a short section of remarks and comments for further reading. Some recent papers on special functions, especially those related to their asymptotic behavior, are mentioned here. All chapters end with long lists of exercises, which include many of the important properties of special functions.

Researchers in mathematics, physics, and engineering who occasionally use special functions will find it a useful reference. It is comparable to the two well-known texts by Lebedev and by Olver. The printing of this book and the layout of the material are also very good. Important formulas on each page are highlighted in boxes for easy reference.