2 Functions

After working through this section, you should be able to:

(a) determine the *image* of a given function;

(b) determine whether a given function is *one-one* and/or *onto*;

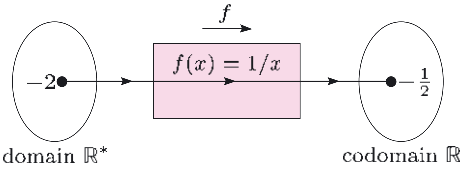
(c) find the *inverse* of a given one-one function;

(d) find the *composite* of two given functions.

2.1 What is a function?

In the previous unit we concentrated on *real* functions—that is, functions whose domains and codomains are subsets of $\mathbb{R}$. You can think of these functions as machines for processing real numbers. For example, the real function defined by $f(x) = 1/x$ can be regarded as a machine that calculates the reciprocals of non-zero real numbers. When 3 is fed into the machine, out comes the number $\frac{1}{3}$; when -2 is fed into the machine, out comes $-\frac{1}{2}$; and so on. Indeed, any real number in the domain $\mathbb{R}^*$ of f can be processed by the machine to produce a real number in the codomain.

Recall from Subsection 1.1 that $\mathbb{R}^*$ denotes the set of non-zero real numbers, $\mathbb{R} - \{0\}$.



Now imagine a machine that accepts an element x from a set A (not necessarily a subset of $\mathbb{R}$), and processes it to produce a unique element $f(x)$ in a set B (again not necessarily a subset of $\mathbb{R}$). By dropping the requirement that the machine processes and produces real numbers, we obtain the following more general definition of a function.

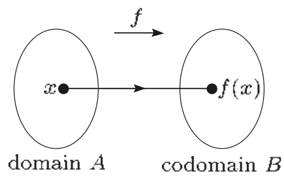
Definitions A **function** f is defined by specifying:

- a set A, called the **domain** of f;
- a set B, called the **codomain** of f;
- a **rule** $x \longmapsto f(x)$ that associates with each element $x \in A$ a unique element $f(x) \in B$.

The element $f(x)$ is the **image** of x under f.

Symbolically, we write

$$\begin{aligned} f : A &\longrightarrow B \\ x &\longmapsto f(x). \end{aligned}$$



We often refer to a function as a **mapping**, and say that f **maps** A to B and x to $f(x)$.

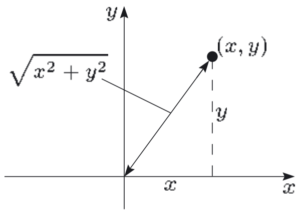
Since the domain A and the codomain B are no longer restricted to be sets of real numbers, we can now study many types of function in addition to the real functions that you met in Unit I1. We present a few examples.

Distance function

Functions of the form $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ can be used to specify quantities associated with points in the plane. For example, the function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \sqrt{x^2 + y^2} \end{aligned}$$

gives the distance of each point (x, y) in the plane from the origin.

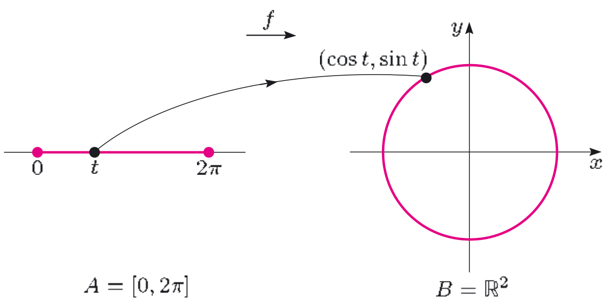


Parametrisations

In Unit I1, Section 5, you saw that functions of the form $f : I \longrightarrow \mathbb{R}^2$, where I is an interval of $\mathbb{R}$, can be used to parametrise curves in the plane. For example, the function

$$\begin{aligned} f : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

is a parametrisation of the unit circle.



Transformations of the plane

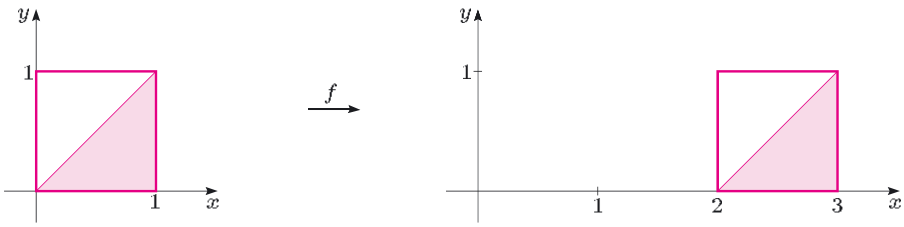
Functions that have a geometric interpretation are often called *transformations*. Such functions include translations, reflections and rotations of the plane. We now present some simple examples. For each one, we give a diagram which shows the effect of the transformation on the square whose vertices are at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$; part of the square is shaded for clarity.

These types of transformation were introduced in Unit I1, Section 1, Frames 10, 18 and 20. You will study more complicated transformations in the Linear Algebra Block.

The transformation

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + 2, y)$$

is a *translation* of the plane that shifts (or translates) each point to the right by 2 units.

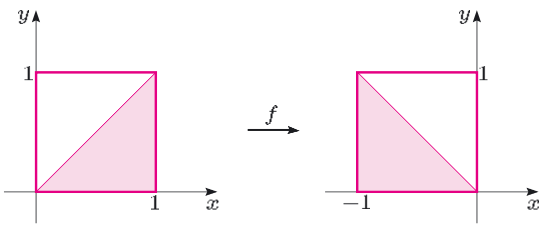


translation 2 units to right

The transformation

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (-x, y)$$

is a *reflection* of the plane in the y-axis.

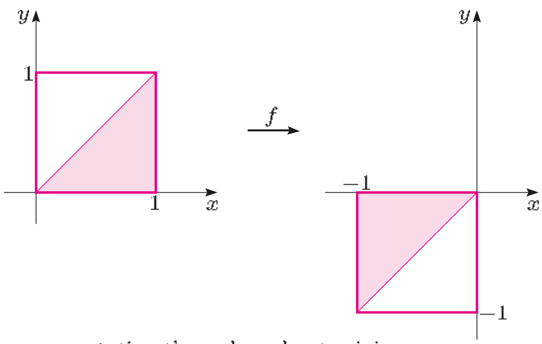


reflection in y-axis

The transformation

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (-x, -y)$$

is a *rotation* of the plane through π about the origin.



rotation through π about origin

Exercise 2.1 For each of the following functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, state whether f is a translation, reflection or rotation of the plane.

(a) $f(x, y) = (x + 2, y + 3)$

(b) $f(x, y) = (x, -y)$

(c) $f(x, y) = (-y, x)$

For simplicity, we write $f(x, y)$ instead of $f((x, y))$.

Functions on finite sets

It is often useful to consider a function whose domain is a *finite* set. For example, we can define a function whose domain is the set

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

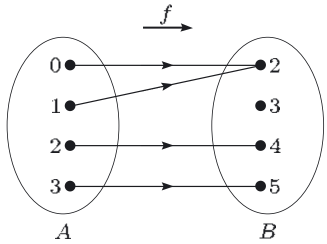
by

$$\begin{aligned} f : A &\longrightarrow A \\ x &\longmapsto 9 - x. \end{aligned}$$

When the domain of a function f has a small number of elements, we can specify the rule of f by listing the image $f(x)$ of each element x in the domain. For example, let $A = \{0, 1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$; then we can define a function $f : A \longrightarrow B$ by the rule

$$f(0) = 2, \quad f(1) = 2, \quad f(2) = 4, \quad f(3) = 5.$$

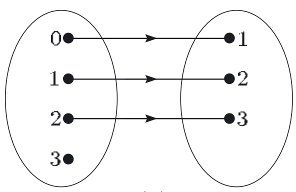
We can represent the behaviour of this function by a diagram, as shown below.



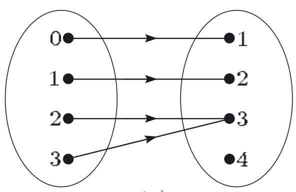
There is exactly *one* arrow *from* each element in the domain, since each element in the domain has exactly *one* image in the codomain. There may be no arrows, one arrow or several arrows going *to* an element in the codomain, since an element in the codomain may not be an image at all, may be an image of exactly one element in the domain, or may be an image of several elements in the domain.

For example, for this function, 3 is not an image at all, 5 is the image of 3 only, and 2 is the image of both 0 and 1.

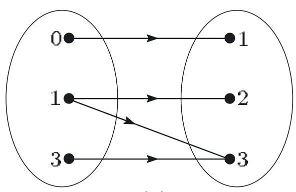
Exercise 2.2 Which of the following diagrams correspond(s) to a function?



(a)



(b)

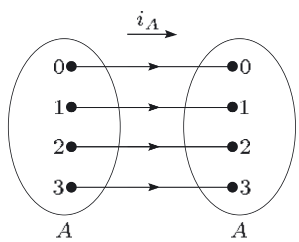


(c)

Identity functions

Associated with any set A, there is a particularly simple function whose domain and codomain are the set A. This is the *identity function* i_A which maps each element of A to itself.

We sometimes omit the subscript A if we do not need to emphasise the set.



For example, let $A = \{0, 1, 2, 3\}$; then the rule of the identity function is

$$i_A(0) = 0, \quad i_A(1) = 1, \quad i_A(2) = 2, \quad i_A(3) = 3.$$

The following definition applies to *any* set A, finite or infinite.

Definition The **identity function** on a set A is the function

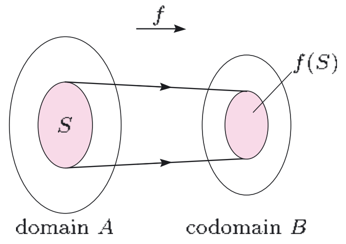
$$i_A : A \longrightarrow A$$
$$x \longmapsto x.$$

2.2 Image of a function

The rule associated with a function tells us how to find the image of any element in the domain. Often, however, we need to consider the images of a whole subset of elements drawn from the domain; for example, in geometry, we frequently wish to consider the effect of a transformation on a plane figure, a subset of $\mathbb{R}^2$.

Definition Given a function $f : A \longrightarrow B$, and a subset S of A, the **image**, or **image set**, of S under f, written $f(S)$, is the set

$$f(S) = \{f(x) : x \in S\}.$$



For example, suppose that S is the square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, and we want to find the image of S under the function

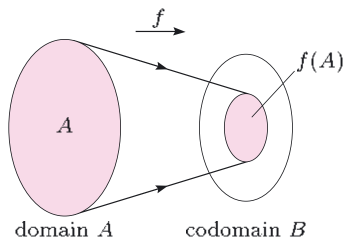
$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + 2, y).$$

This function is the translation of the plane which moves each point (x, y) to the right by 2. The image of S is therefore the square with vertices at $f(0, 0) = (2, 0)$, $f(1, 0) = (3, 0)$, $f(1, 1) = (3, 1)$ and $f(0, 1) = (2, 1)$.

Sometimes we wish to consider the image of the *whole domain* of a function: this is referred to as the image, or image set, of the *function*.

Definition The **image**, or **image set**, of a function $f : A \longrightarrow B$ is the set

$$f(A) = \{f(x) : x \in A\}.$$



The image of a function is a subset of its codomain. It need not be *equal* to the codomain because there may be some elements of the codomain that are not images of elements in the domain.

When the domain of a function f has a small number of elements, we can find the image of f by finding the image of each element in the domain, and listing them to form a set.

Example 2.1 Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Find the image of the function

$$f : A \longrightarrow A$$
$$x \longmapsto \left[\tfrac{1}{2}x\right].$$

Recall that $[x]$, the *integer part* of x, is the largest integer that is less than or equal to x. For example, $[3.7] = 3$, $[3] = 3$ and $[-3.7] = -4$.

Solution The images of the elements of A are

$$f(0)=0,\quad f(1)=0,\quad f(2)=1,\quad f(3)=1,\quad f(4)=2,$$
$$f(5)=2,\quad f(6)=3,\quad f(7)=3,\quad f(8)=4,\quad f(9)=4.$$

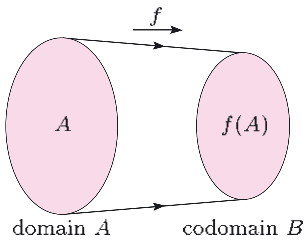
So the image of f is $f(A)=\{0,1,2,3,4\}$. ■

Exercise 2.3 Let $A=\{0,1,2,3,4,5,6,7,8,9\}$.

Find the image of the function

$$\begin{aligned} f : A &\longrightarrow A \\ x &\longmapsto 9-x. \end{aligned}$$

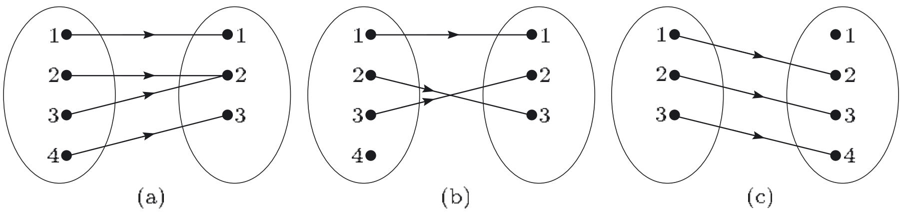
In Exercise 2.3 you should have found that the image and the codomain coincide. In other words, each element of the codomain is the image of an element in the domain. A function with this property is said to be *onto*.



f is onto: $f(A)=B$

Definition A function $f : A \longrightarrow B$ is **onto** if $f(A)=B$.

Exercise 2.4 Which of the following diagrams correspond(s) to an onto function?



Example 2.2 For each of the following functions, find its image and determine whether it is onto.

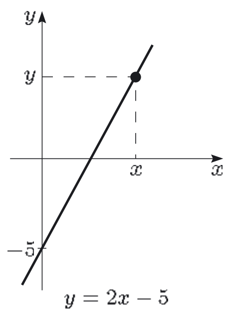
Some texts refer to an onto function as a *surjective* function.

(a) $\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x-5 \end{aligned}$ (b) $\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$ (c) $\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+1,y+2) \end{aligned}$

(d) $\begin{aligned} f : A &\longrightarrow A, \quad \text{where } A=\{0,1,2,3,4,5,6,7,8,9\} \\ x &\longmapsto \left[\tfrac{1}{2}x\right] \end{aligned}$

Solution

(a) The sketch of the graph of f shown in the margin suggests that the image of f is the whole of $\mathbb{R}$. To confirm this, we prove algebraically that $f(\mathbb{R})=\mathbb{R}$.



We know that $f(\mathbb{R})\subseteq\mathbb{R}$, so we must show that $f(\mathbb{R})\supseteq\mathbb{R}$.

Let y be an arbitrary element in $\mathbb{R}$. We must show that $y\in f(\mathbb{R})$; that is, there exists an element x in the domain $\mathbb{R}$ such that

$$f(x)=y;\quad \text{that is,}\quad 2x-5=y.$$

Rearranging this equation, we obtain

$$x=\frac{y+5}{2}.$$

This is in $\mathbb{R}$ and satisfies $f(x)=y$, as required. Thus $f(\mathbb{R})\supseteq\mathbb{R}$.

In this solution, we use $\supseteq$, rather than $\subseteq$, so that the image is always on the left and the codomain on the right.

Since $f(\mathbb{R})\subseteq\mathbb{R}$ and $f(\mathbb{R})\supseteq\mathbb{R}$, it follows that $f(\mathbb{R})=\mathbb{R}$, so the image of f is indeed $\mathbb{R}$.

The codomain of f is also $\mathbb{R}$, so f is onto.

(b) The sketch of the graph of f shown in the margin suggests that the image of f is $[0, \infty)$. We now prove algebraically that $f(\mathbb{R}) = [0, \infty)$.

Let x be an arbitrary element in the domain $\mathbb{R}$; then $f(x) = x^2 \geq 0$, so $f(\mathbb{R}) \subseteq [0, \infty)$.

We must show that $f(\mathbb{R}) \supseteq [0, \infty)$.

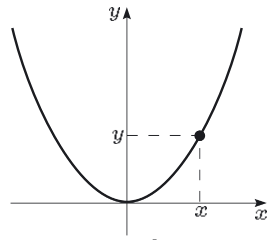
Let y be an arbitrary element in $[0, \infty)$. We must show that there exists an element x in the domain $\mathbb{R}$ such that

$$f(x) = y; \quad \text{that is,} \quad x^2 = y.$$

Now $x = \sqrt{y}$ is in $\mathbb{R}$ (since $y \geq 0$) and satisfies $f(x) = y$, as required. Thus $f(\mathbb{R}) \supseteq [0, \infty)$.

Since $f(\mathbb{R}) \subseteq [0, \infty)$ and $f(\mathbb{R}) \supseteq [0, \infty)$, it follows that $f(\mathbb{R}) = [0, \infty)$, so the image of f is $[0, \infty)$, as expected.

The interval $[0, \infty)$ is not the whole of the codomain, so f is not onto.



$y = x^2$

Alternatively, we could choose the real number $x = -\sqrt{y}$, which also satisfies $f(x) = y$.

(c) This function is a translation of the plane that shifts each point to the right by 1 unit and up by 2 units. This suggests that $f(\mathbb{R}^2) = \mathbb{R}^2$.

We know that $f(\mathbb{R}^2) \subseteq \mathbb{R}^2$, so we must show that $f(\mathbb{R}^2) \supseteq \mathbb{R}^2$.

Let (x', y') be an arbitrary element in the codomain $\mathbb{R}^2$. We must show that there exists an element (x, y) in the domain $\mathbb{R}^2$ such that

$$f(x, y) = (x', y'); \quad \text{that is,} \quad x' = x + 1, \quad y' = y + 2.$$

Rearranging these two equations, we obtain

$$x = x' - 1, \quad y = y' - 2.$$

With these values, $(x, y) \in \mathbb{R}^2$ and $f(x, y) = (x', y')$, as required. Thus $f(\mathbb{R}^2) \supseteq \mathbb{R}^2$.

Since $f(\mathbb{R}^2) \subseteq \mathbb{R}^2$ and $f(\mathbb{R}^2) \supseteq \mathbb{R}^2$, it follows that $f(\mathbb{R}^2) = \mathbb{R}^2$, so the image of f is $\mathbb{R}^2$, as expected.

The codomain of f is also $\mathbb{R}^2$, so f is onto.

(d) In Example 2.1, we showed that the image of this function is $\{0, 1, 2, 3, 4\}$. This is not the whole of the codomain, so f is not onto. ■

Exercise 2.5 For each of the following functions, find its image and determine whether it is onto.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$
$\quad x \longmapsto 1 + x^2$

(b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$\quad (x, y) \longmapsto (x, -y)$

2.3 Inverse functions

Given a function

$$f : A \longrightarrow B$$
$$x \longmapsto f(x),$$

it is sometimes possible to define an *inverse function* that 'undoes' the effect of f by mapping each image element $f(x)$ back to the element x whose image it is. For example, a rotation can be 'undone' by a rotation in the opposite direction.

However, consider the function

$$f : A \longrightarrow A, \quad \text{where } A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$
$$x \longmapsto \left[\tfrac{1}{2}x\right].$$

We know that $f(2) = f(3) = 1$, and so a function that 'undoes' the effect of f must map the number 1 to the number 2 *and* to the number 3, which is impossible. Thus, in this case, no inverse function exists. This function f is an example of a function that is *many-one*. A many-one function does not have an inverse function.

Definitions A function $f : A \longrightarrow B$ is **one-one** if each element of $f(A)$ is the image of exactly one element of A; that is,

$$\text{if } x_1, x_2 \in A \text{ and } f(x_1) = f(x_2), \quad \text{then } x_1 = x_2.$$

A function that is not one-one is **many-one**.

Some texts refer to a one-one function as an *injective* function.

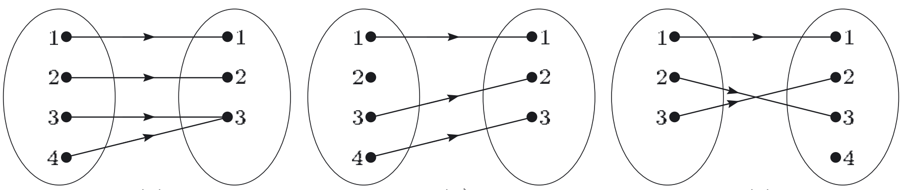
We write $x_1, x_2 \in A$ as shorthand for $x_1 \in A$ and $x_2 \in A$.

Remark Thus a function f is one-one if it maps distinct elements in the domain A to distinct elements in the image $f(A)$.

The term 'one-one' is often read as 'one to one'; similarly 'many-one' is often read as 'many to one'.

To prove that a function f is *not* one-one, it is sufficient to find a single *counter-example*—that is, a pair of *distinct* elements in the domain A with the *same* image in $f(A)$.

Exercise 2.6 Which of the following diagrams correspond(s) to a one-one function?



(a) (b) (c)

Example 2.3 Determine which of the following functions are one-one.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 2x - 5$

(b) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$

(c) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \longmapsto (x + 1, y + 2)$

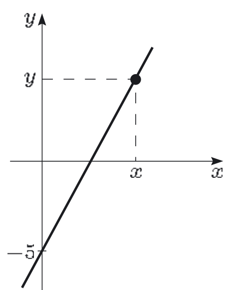
Solution

(a) The graph suggests that f is one-one. We now prove this algebraically. Suppose that $f(x_1) = f(x_2)$; then

$$2x_1 - 5 = 2x_2 - 5,$$

so $2x_1 = 2x_2$, and hence $x_1 = x_2$.

Thus f is one-one.

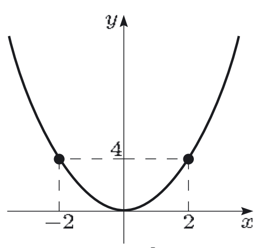


$y = 2x - 5$

(b) The graph suggests that f is not one-one. To show that f is not one-one, we just need to find two distinct points in the domain of f with the same image. For example,

$$f(2) = f(-2) = 4,$$

so f is not one-one.



$y = x^2$

(c) This function is a translation of the plane, and so we expect it to be one-one. We now prove this algebraically.

Suppose that $f(x_1, y_1) = f(x_2, y_2)$; then

$$(x_1 + 1, y_1 + 2) = (x_2 + 1, y_2 + 2).$$

Thus

$$x_1 + 1 = x_2 + 1 \quad \text{and} \quad y_1 + 2 = y_2 + 2,$$

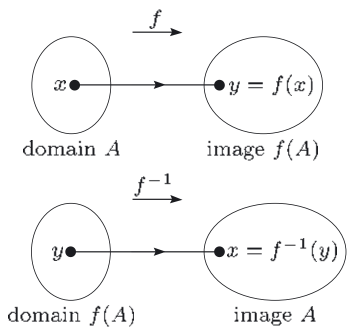
so $x_1 = x_2$ and $y_1 = y_2$.

Hence $(x_1, y_1) = (x_2, y_2)$, so f is one-one. ■

Exercise 2.7 Determine which of the following functions is one-one.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 1 + x^2$ (b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \longmapsto (x, -y)$

For a one-one function $f : A \longrightarrow B$, we have the situation illustrated in the margin. Each element y in $f(A)$ is the image of a unique element x in A, and so we can reverse the arrows to obtain the *inverse function* f^{-1}, with domain $f(A)$ and image A, which maps y back to x.



Definition Let $f : A \longrightarrow B$ be a one-one function. Then f has an **inverse function** $f^{-1} : f(A) \longrightarrow A$, with rule

$$f^{-1}(y) = x, \quad \text{where } y = f(x).$$

A function $f : A \longrightarrow B$ that is both one-one and onto has an inverse function $f^{-1} : B \longrightarrow A$. Such a function f is said to be a **one-one correspondence**, or a **bijection**, between the sets A and B.

Example 2.4 For each of the following functions, determine whether f has an inverse function f^{-1}; if it exists, find it.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 2x - 5$ (b) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$ (c) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \longmapsto (x + 1, y + 2)$

(d) $f : A \longrightarrow A$, $x \longmapsto \left[\frac{1}{2}x\right]$ where $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Solution

(a) In Example 2.3(a), we showed that f is one-one, so f has an inverse function.

In Example 2.2(a), we showed that the image of f is $\mathbb{R}$ and that, for each y in the image $\mathbb{R}$, we have

$$y = f\left(\frac{y + 5}{2}\right).$$

So f^{-1} is the function

$$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}, \quad y \longmapsto \frac{y + 5}{2}.$$

This definition can be expressed in terms of x as

$$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x + 5}{2}.$$

It does not matter whether the definition of f^{-1} is expressed in terms of x or y, but it is more usual to use x in the definition of a real function.

(b) In Example 2.3(b), we showed that f is not one-one, so f does not have an inverse function.

(c) In Example 2.3(c), we showed that f is one-one, so f has an inverse function.

In Example 2.2(c), we showed that the image of f is $\mathbb{R}^2$ and that, for each (x', y') in the image $\mathbb{R}^2$, we have

$$(x', y') = f(x' - 1, y' - 2).$$

So f^{-1} is the function

$$\begin{aligned} f^{-1} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x', y') &\longmapsto (x' - 1, y' - 2). \end{aligned}$$

Geometrically, f is the translation of the plane that shifts each point to the right by 1 unit and up by 2 units. Since f^{-1} undoes the effect of f, the inverse f^{-1} is the translation that shifts each point to the left by 1 unit and down by 2 units.

This definition can be expressed in terms of x and y as

$$\begin{aligned} f^{-1} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x - 1, y - 2). \end{aligned}$$

(d) At the beginning of this subsection, we showed that f is not one-one, so f does not have an inverse function. ■

Exercise 2.8 For each of the following functions, determine whether f has an inverse function f^{-1}; if it exists, find f^{-1}.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 1 + x^2$ (b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \longmapsto (x, -y)$ (c) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto 8x + 3$

Hint: For parts (a) and (b), use your answers to Exercises 2.5 and 2.7.

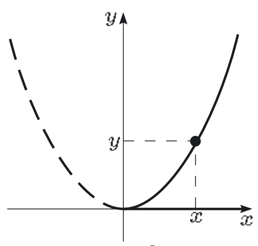
Restrictions

When given a function $f : A \longrightarrow B$, it is often convenient to restrict attention to the behaviour of f on some subset C of A. For example, consider the function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2. \end{aligned}$$

This function is not one-one and so does not have an inverse function. However, if the domain of f is replaced by the set $C = [0, \infty)$, then we obtain a related function

$$\begin{aligned} g : C &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2. \end{aligned}$$



$y = x^2$

The rule is the same as for f, but the domain is 'restricted' to produce a new function g that is one-one and so has an inverse.

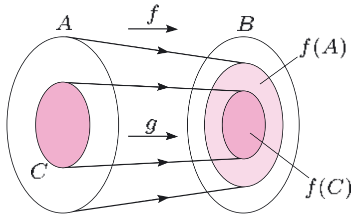
The function g is an example of a *restriction* of f in the sense that $g(x) = f(x)$ for all x in the domain of g.

More generally, we define a restriction as follows.

Definition Let $f : A \longrightarrow B$ and let C be a subset of the domain A. Then the function $g : C \longrightarrow B$ defined by

$$g(x) = f(x), \quad \text{for } x \in C,$$

is the **restriction** of f to C.



g is the restriction of f to C

Exercise 2.9 Let f be the function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto |x|. \end{aligned}$$

Write down a restriction of f that is one-one.

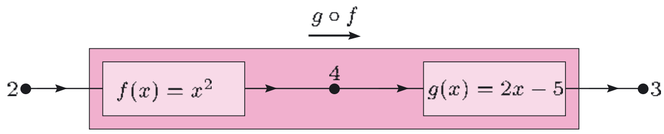
2.4 Composite functions

Earlier, we described how a function may be regarded as a machine that processes elements in the domain to produce elements in the codomain. Now suppose that two such machines are linked together, so that the elements emerging from the first machine are fed into the second machine for further processing. The overall effect is to create a new 'composite' machine that corresponds to a so-called *composite* function.

Consider the real functions

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned} \quad \text{and} \quad \begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x - 5. \end{aligned}$$

When the machines for f and g are linked together so that objects are first processed by f and then by g, we obtain the 'composite' machine illustrated by the dark blue box in the diagram below.



For example, when 2 is fed into the machine, it is first squared by f to produce the number 4, and then 4 is processed by g to give the number $(2 \times 4) - 5 = 3$.

Similarly, when an arbitrary real number x is fed into the machine, it is first processed by f to give the real number x^2. Since the domain of g is the whole of $\mathbb{R}$, the number x^2 can then be processed by g to give $2x^2 - 5$. Thus, overall, the composite machine corresponds to a function, denoted by $g \circ f$, whose rule is

$$(g \circ f)(x) = g(f(x)) = 2x^2 - 5$$

Remember that $g \circ f$ means f first, then g.

and whose domain is $\mathbb{R}$.

In general, if $f : A \longrightarrow B$ and $g : B \longrightarrow C$, then we can form the *composite function*

Here the domain of the function $g : B \longrightarrow C$ is the same as the codomain of the function $f : A \longrightarrow B$.

$$\begin{aligned} g \circ f : A &\longrightarrow C \\ x &\longmapsto g(f(x)). \end{aligned}$$

Exercise 2.10 Let f and g be the functions

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto -x \end{aligned} \quad \text{and} \quad \begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 3x + 1. \end{aligned}$$

Determine the composite functions

(a) $g \circ f$, (b) $f \circ g$.

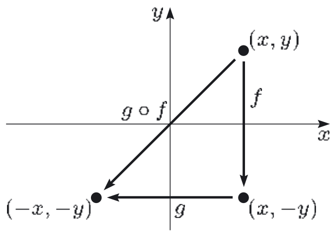
The composite functions $g \circ f$ and $f \circ g$ are not equal in general, as you saw in the above exercise.

Composite functions have many uses in mathematics. In geometry, they are frequently used to examine the effect of one transformation of the plane followed by another.

Suppose, for example, that f and g are the reflections of the plane in the x-axis and y-axis respectively:

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, -y) \end{aligned} \quad \text{and} \quad \begin{aligned} g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-x, y). \end{aligned}$$

The composite function $g \circ f$ describes the overall effect of first reflecting in the x-axis (changing the sign of y) and then reflecting in the y-axis (changing the sign of x).



The rule of $g \circ f$ is

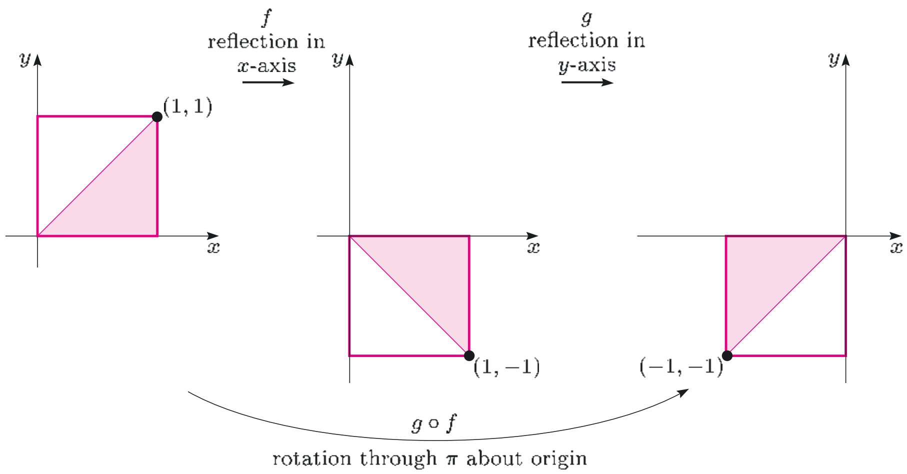
$$\begin{aligned} (g \circ f)(x, y) = g(f(x, y)) &= g(x, -y) \\ &= (-x, -y). \end{aligned}$$

Thus $g \circ f$ is the function

$$\begin{aligned} g \circ f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-x, -y), \end{aligned}$$

which rotates the plane through π about the origin.

For example, a square is transformed as follows.



Exercise 2.11 Determine the composite function $f \circ g$ of the following transformations of the plane:

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, -y) \end{aligned} \quad \text{and} \quad \begin{aligned} g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-x, y). \end{aligned}$$

So far, we have considered the composite function $g \circ f$ only when the domain of the function $g : B \longrightarrow C$ is the same as the codomain of the function $f : A \longrightarrow B$. We can, however, form the composite function $g \circ f$ when g and f are *any* two functions.

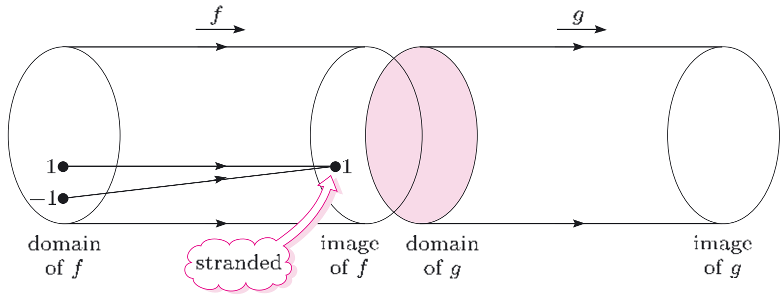
For example, consider the functions

$$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{and} \qquad g : \mathbb{R} - \{1\} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2 \qquad\qquad x \longmapsto \frac{1}{x-1}.$$

Here the domain of g is not equal to the codomain of f, but we can still consider the composite function $g \circ f$, with the rule

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \frac{1}{x^2 - 1}.$$

In this case, the domain of $g \circ f$ cannot be $\mathbb{R}$ (the domain of f) since 1 and -1 are both mapped by f to the number 1, which is not in the domain of g; this means that no further processing is possible.



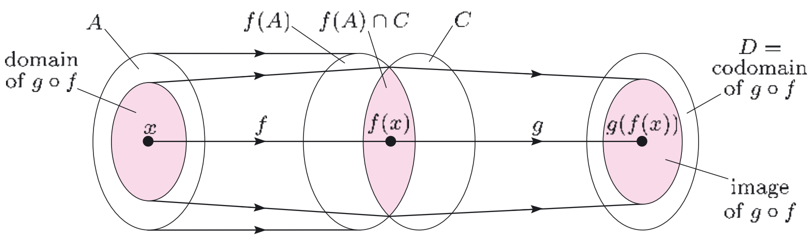
To overcome this difficulty, we take the domain of $g \circ f$ to be the difference $\mathbb{R} - \{1, -1\}$. So the composite function $g \circ f$ is

$$g \circ f : \mathbb{R} - \{1, -1\} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{1}{x^2 - 1}.$$

The general situation is illustrated below. Given any two functions $f : A \longrightarrow B$ and $g : C \longrightarrow D$, the rule of the composite function $g \circ f$ is

$$(g \circ f)(x) = g(f(x)).$$

The domain of $g \circ f$ consists of all those elements x in A for which $f(x)$ lies in C (the domain of g). The codomain of $g \circ f$ is defined to be D (the codomain of g).



Definition Let $f : A \longrightarrow B$ and $g : C \longrightarrow D$ be any two functions; then the **composite function** $g \circ f$ has

domain	$\{x \in A : f(x) \in C\}$,
codomain	D,
rule	$(g \circ f)(x) = g(f(x))$.

This definition allows us to consider the composite of *any* two functions, although in some cases the domain may turn out to be the empty set $\varnothing$. Some authors insist upon $f(A) \subseteq C$ as a condition for $g \circ f$ to exist.

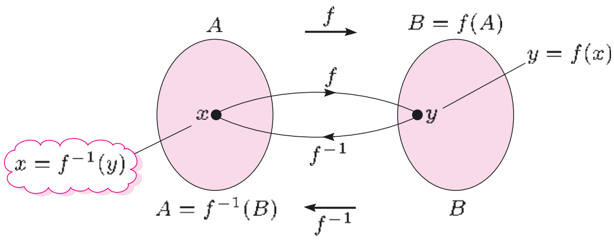
In the example on page 32, the domain of $g \circ f$ is just the set of values for which the rule of $g \circ f$ is defined. This is not the case in the following exercise.

Exercise 2.12 Determine the composite function $g \circ f$ for the following functions f and g:

$$f : [-1, 1] \longrightarrow \mathbb{R} \quad x \longmapsto 3x + 1 \quad \text{and} \quad g : \mathbb{R} - \{-2\} \longrightarrow \mathbb{R} \quad x \longmapsto \frac{3}{x+2}.$$

Composites and inverses

Suppose that $f : A \longrightarrow B$ is a one-one and onto function. Then f has an inverse function $f^{-1} : B \longrightarrow A$. We can therefore consider the effect that the composite function $f^{-1} \circ f : A \longrightarrow A$ has on an arbitrary element x in A. First, f maps x to an element $y = f(x)$ in $f(A)$. Then f^{-1} 'undoes' the effect of f and maps y back to x. Overall, the effect of $f^{-1} \circ f$ is to leave x fixed: that is, $(f^{-1} \circ f)(x) = x$. Since x is an arbitrary element of A, it follows that $f^{-1} \circ f$ fixes all the elements of A. In other words, $f^{-1} \circ f = i_A$.



Similar arguments can be used to show that $f \circ f^{-1} = i_B$. So, if $f : A \longrightarrow B$ has an inverse function $f^{-1} : B \longrightarrow A$, then

$$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B.$$

The converse of this statement is also true: if a function $g : B \longrightarrow A$ satisfies

$$g \circ f = i_A \quad \text{and} \quad f \circ g = i_B,$$

then g is the inverse function of f. We shall prove this after Exercise 2.13. It leads to the following strategy.

Strategy 2.1 To show that the function $g : B \longrightarrow A$ is the inverse function of the function $f : A \longrightarrow B$.

1. Show that $f(g(x)) = x$ for each $x \in B$; that is, $f \circ g = i_B$.
2. Show that $g(f(x)) = x$ for each $x \in A$; that is, $g \circ f = i_A$.

In practice, we can sometimes use Strategy 2.1 as an alternative way of *finding* an inverse function. We make an inspired guess at the function, and use Strategy 2.1 to check that our guess is correct.

Example 2.5 Find the inverse of the function

$$g : \mathbb{R} \longrightarrow \mathbb{R} \quad x \longmapsto \tfrac{1}{2}x.$$

Solution We guess that the inverse function is

$$f : \mathbb{R} \longrightarrow \mathbb{R} \quad x \longmapsto 2x.$$

We use Strategy 2.1 to check that our guess is correct.

1. For each $x \in \mathbb{R}$, we have

$$f(g(x)) = f(\tfrac{1}{2}x) = 2 \times \tfrac{1}{2}x = x;$$

that is, $f \circ g = i_{\mathbb{R}}$.

2. For each $x \in \mathbb{R}$, we have

$$g(f(x)) = g(2x) = \tfrac{1}{2} \times 2x = x;$$

that is, $g \circ f = i_{\mathbb{R}}$.

Since $f \circ g = i_{\mathbb{R}}$ and $g \circ f = i_{\mathbb{R}}$, it follows that g is the inverse function of f. ■

Exercise 2.13 Use Strategy 2.1 to show that

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x - 3 \end{aligned}$$

is the inverse function of

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x + 3. \end{aligned}$$

We end this section by proving, as promised, that if the functions $f : A \longrightarrow B$ and $g : B \longrightarrow A$ satisfy

$$g \circ f = i_A \quad \text{and} \quad f \circ g = i_B,$$

then g is the inverse function of f. That is, we have to show that if the two steps of Strategy 2.1 hold, then f has an inverse function, and the inverse function is equal to g. Suppose that the two steps of Strategy 2.1 hold.

First we show that f is one-one. Suppose that $f(x_1) = f(x_2)$; then

$$g(f(x_1)) = g(f(x_2)),$$

so, by step 2, $x_1 = x_2$. Thus f is one-one and so it has an inverse function f^{-1}.

Now we find the image of f. We know that the image of f is a subset of its codomain B; we shall show that it is equal to B by showing that every element y of B is the image under f of some element in A. Suppose that $y \in B$. Then, by step 1,

$$f(g(y)) = y;$$

that is, y is the image under f of the element $g(y)$, as required. Thus the image of f is B (that is, f is onto), and so f^{-1} has domain B.

We now know that each of the functions f^{-1} and g has domain B and codomain A. To show that they are equal, it remains to show that $g(y) = f^{-1}(y)$ for each element y of B. Let y be an arbitrary element of B. Then $y = f(x)$ for some element x of A. So

$$f^{-1}(y) = x,$$

and, by step 2,

$$g(y) = g(f(x)) = x.$$

Hence f^{-1} and g are indeed equal functions.

Further exercises

Exercise 2.14 For each of the following transformations $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, state whether f is a translation, reflection or rotation of the plane.

(a) $f(x, y) = (y, -x)$ (b) $f(x, y) = (x - 2, y + 1)$

Exercise 2.15 Draw a diagram showing the image of T, the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(1, 1)$, under each of the functions f of Exercise 2.14.

Exercise 2.16 For each of the following functions, find its image and determine whether it is onto.

(a) $$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-y, x) \end{aligned}$$

(b) $$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 7 - 3x \end{aligned}$$

(c) $$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 - 4x + 3 \end{aligned}$$

(d) $$\begin{aligned} f : [0, 1] &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x + 3 \end{aligned}$$

Exercise 2.17 Determine which of the functions in Exercise 2.16 are one-one.

Exercise 2.18 Determine which of the functions in Exercise 2.16 has an inverse, and find the inverse f^{-1} for each one which does.

Exercise 2.19 Determine the composite $f \circ g$ for each of the following pairs of functions f and g.

(a) $$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 7 - 3x \end{aligned} \quad \text{and} \quad \begin{aligned} g : \mathbb{R} - \{2, -2\} &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{x^2 - 4}. \end{aligned}$$

(b) $$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-y, x) \end{aligned} \quad \text{and} \quad \begin{aligned} g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (y, x). \end{aligned}$$