4 Two identities

After working through this section, you should be able to:

(a) understand and use the Binomial Theorem;
(b) understand and use the Geometric Series Identity;
(c) understand and use the Polynomial Factorisation Theorem.

An **identity** is an equation involving variables which is true for all possible values of the variables. You will already be familiar with many basic identities, such as

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad a^2 - b^2 = (a-b)(a+b).$$

Some texts use the symbol $\equiv$ to denote an identity, but we shall not do so.

These identities are particular cases of more general identities that we shall use extensively later in the course. In this section we state and prove these key identities, using some of the techniques described earlier in the unit.

4.1 The Binomial Theorem

A striking mathematical pattern appears when we expand expressions of the form $(a+b)^n$ for $n = 1, 2, \ldots$:

$$(a+b)^1 = a^1 + b^1,$$
$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = (a+b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a+b)^4 = (a+b)(a^3 + 3a^2b + 3ab^2 + b^3) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

and so on. The coefficients that appear in these expansions can be arranged as a triangular table, in which 1s appear on the left and right edges, and the remaining entries can be generated by using the rule that each inner entry is the sum of the two nearest entries in the row above.

For example, $10 = 4 + 6$.

$$\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & & \vdots & & & & &
\end{array}$$

The 1 at the top corresponds to $n = 0$:
$$(a+b)^0 = 1.$$

This table is known as *Pascal's triangle*, after the French mathematician, physicist and theologian Blaise Pascal (1623–1662), although it appeared several hundred years earlier in a book by the Chinese mathematician Chu Shih-Chieh.

We can calculate any coefficient in Pascal's triangle directly, instead of from two coefficients in the row above, because the coefficients in the row corresponding to $(a+b)^n$ are given by

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

For example, the fourth coefficient in the row corresponding to $(a+b)^5$ is given by
$$\binom{5}{3} = \frac{5!}{3!\,2!} = 10.$$

The expression $\binom{n}{k}$ was introduced in Subsection 1.5, where we saw that it gives the number of different k-element subsets of an n-element set.

Example 1.5 shows why the numbers $\binom{n}{k}$ satisfy the rule described above for generating Pascal's triangle.

Here is the general formula for the expansion of $(a+b)^n$. It is our first key identity.

Theorem 4.1 Binomial Theorem

Let $a, b \in \mathbb{R}$ and let n be a positive integer. Then

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n} b^n.$$

Note that

$$\binom{n}{0} = \binom{n}{n} = 1,$$

since $0! = 1$.

Proof The shortest way of proving this result is to note that $(a+b)^n$ is the product of n brackets:

$$(a+b)^n = (a+b) \times (a+b) \times \cdots \times (a+b).$$

When this product is multiplied out, we find that each term of the form $a^{n-k}b^k$ arises by choosing the variable a from $n-k$ of the brackets and the variable b from the remaining k brackets. Thus the coefficient of $a^{n-k}b^k$ is equal to the number of ways of choosing a subset of $n-k$ brackets (or, equivalently, a subset of k brackets) from the set of n brackets, and this is precisely $\binom{n}{k}$, as required. ■

Note the following important special case of Theorem 4.1, obtained by taking $a = 1$ and $b = x$:

$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{k} x^k + \cdots + \binom{n}{n} x^n \\ &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n.\end{aligned}$$

Example 4.1 Expand $(2+3x)^5$.

Solution Using the Binomial Theorem with $n = 5$, $a = 2$ and $b = 3x$, we obtain

$$\begin{aligned}(2+3x)^5 &= \binom{5}{0} 2^5 + \binom{5}{1} 2^4(3x) + \binom{5}{2} 2^3(3x)^2 \\ &\quad + \binom{5}{3} 2^2(3x)^3 + \binom{5}{4} 2(3x)^4 + \binom{5}{5} (3x)^5 \\ &= 2^5 + 5 \times 2^4(3x) + 10 \times 2^3(3x)^2 \\ &\quad + 10 \times 2^2(3x)^3 + 5 \times 2(3x)^4 + (3x)^5 \\ &= 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5.\end{aligned}$$ ■

Exercise 4.1 Find the coefficient of:

(a) a^5b^4 in the expansion of $(a+b)^9$;

(b) x^4 in the expansion of $(1+2x)^5$.

Many identities can be obtained as special cases of the Binomial Theorem by choosing particular values for the variables a and b.

Example 4.2 Deduce from the Binomial Theorem that

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}, \quad \text{for } n \geq 1.$$

Solution Taking $a = 1$ and $b = 1$ in the statement of the Binomial Theorem, we obtain $(1+1)^n = 2^n$ on the left-hand side, whereas all the powers of a and b on the right-hand side are equal to 1. This gives the required identity. ■

In Subsection 1.5 we proved that a set with n elements has 2^n subsets. The identity in Example 4.2 provides an alternative proof of this fact. Since $\binom{n}{k}$ gives the number of k-element subsets of a set with n elements, the sum

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}$$

gives the total number of subsets (of *all* sizes) of a set with n elements. By the identity in Example 4.2, this sum is equal to 2^n.

Exercise 4.2

(a) Use the Binomial Theorem to obtain an expansion for $(a - b)^n$.

(b) Write down the identity you obtain by taking $a = 1$ and $b = 1$ in part (a), and check that this identity is true for $n = 4$.

It is a remarkable fact that there is a version of the Binomial Theorem which holds when n is a real number, rather than just a positive integer. This general version, which involves an infinite series, was used by Isaac Newton, but a correct proof was not given until the early 19th century by the Norwegian mathematician Niels Abel.

A proof of this infinite version is given later in the course.

4.2 The Geometric Series Identity

Expressions of the form $a^n - b^n$ occur often in mathematics, and we can factorise them in the following simple manner:

$$a^2 - b^2 = (a-b)(a+b),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$
$$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3),$$

and so on. The following general result can be proved by multiplying out the expression on the right-hand side.

Theorem 4.2 Geometric Series Identity

Let $a, b \in \mathbb{R}$ and let n be a positive integer. Then

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

Exercise 4.3

(a) Write down the Geometric Series Identity in full for the case $n = 5$.

(b) Use the Geometric Series Identity to show that

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}),$$

where $a, b \in \mathbb{R}$ and n is an *odd* positive integer. Write down this identity in full for the case $n = 5$.

Theorem 4.2 has some useful consequences. In particular, it includes as a special case the following formula for the sum of a finite geometric series with initial term a, common ratio r, and n terms.

This explains the name of Theorem 4.2.

Corollary Sum of a finite geometric series

Let $a, r \in \mathbb{R}$ and let n be a positive integer. Then

$$a + ar + ar^2 + \cdots + ar^{n-1} = \begin{cases} a\left(\dfrac{1-r^n}{1-r}\right), & \text{if } r \neq 1, \\ na, & \text{if } r = 1. \end{cases}$$

Recall that a *corollary* is a consequence of a theorem, proved by a short additional argument.

Proof For the case $r \neq 1$, we need to show that

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r};$$

that is,

$$1 - r^n = (1-r)(1 + r + r^2 + \cdots + r^{n-1}).$$

But this follows from the statement of Theorem 4.2 with $a = 1$ and $b = r$.

When $r = 1$, the required identity is evident, since the left-hand side then consists of n terms all equal to a. ■

Exercise 4.4 Find the sum of the following finite geometric series:

$$1 + 1/2 + 1/4 + \cdots + 1/2^{n-1}.$$

Theorem 4.2 can also be used to give a short proof of a useful result which helps us to factorise polynomials.

A polynomial in x of degree n is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$.

Theorem 4.3 Polynomial Factorisation Theorem

Let $p(x)$ be a polynomial of degree n, and let $\alpha \in \mathbb{R}$. Then $p(\alpha) = 0$ if and only if

$$p(x) = (x - \alpha)q(x), \tag{4.1}$$

where q is a polynomial of degree $n - 1$.

Proof First, we prove the 'if' part.

Here we prove an equivalence by proving separately the two implications that it comprises.

If equation (4.1) holds, then $p(\alpha) = (\alpha - \alpha)q(\alpha) = 0$.

Next, we prove the 'only if' part.

Suppose that $p(\alpha) = 0$. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$. Since $p(\alpha) = 0$,

$$\begin{aligned} p(x) &= p(x) - p(\alpha) \\ &= (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\ &\quad - (a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0) \\ &= a_n(x^n - \alpha^n) + a_{n-1}(x^{n-1} - \alpha^{n-1}) + \cdots + a_1(x - \alpha), \end{aligned}$$

since the constant terms a_0 cancel. Now, by Theorem 4.2, each of the bracketed expressions in this last expression has factor $x - \alpha$, so $p(x)$ is the product of $x - \alpha$ and a polynomial of the form

$$q(x) = a_n x^{n-1} + \cdots,$$

which has degree $n - 1$. ■

Although the proof of Theorem 4.3 could be used to find the polynomial $q(x)$, it is usually easier to find this polynomial by comparing coefficients, once we know that $x - \alpha$ is a factor.

Example 4.3 Show that $x - 2$ is a factor of the cubic polynomial

$$p(x) = x^3 + x^2 - x - 10,$$

and find the corresponding factorisation of $p(x)$.

Solution First, we evaluate $p(2)$:

$$p(2) = 2^3 + 2^2 - 2 - 10 = 8 + 4 - 2 - 10 = 0.$$

Therefore, by the Polynomial Factorisation Theorem, $p(x)$ has the factor $x - 2$. By comparing the coefficients of x^3, and by comparing the constant terms, we obtain

$$x^3 + x^2 - x - 10 = (x - 2)(x^2 + cx + 5), \quad \text{for some number } c.$$

We obtain this by noting that the coefficient of x^2 in the quadratic expression on the right-hand side must be 1 in order to give 1 as the coefficient of x^3 on the left-hand side. Similarly, the constant term in the quadratic expression must be 5 to give the constant term -10 on the left-hand side.

The coefficient of x^2 is 1 on the left-hand side, and $-2 + c$ on the right-hand side, so $c = 3$, which gives

$$x^3 + x^2 - x - 10 = (x - 2)(x^2 + 3x + 5). \quad ■$$

Exercise 4.5 For what value of c is $x + 3$ a factor of

$$p(x) = x^3 + cx^2 + 6x + 36\,?$$

The following result about polynomial factorisation can be proved by applying the Polynomial Factorisation Theorem repeatedly, although we omit the details here. We have taken the coefficient of the highest power of x to be 1, for simplicity. The **roots** of a polynomial $p(x)$ are the solutions of the equation $p(x) = 0$.

The roots of a polynomial are also known as its *zeros*.

Corollary Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and suppose that $p(x)$ has n distinct real roots, $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then

$$p(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n). \tag{4.2}$$

In fact, as you will see in Unit I3, *every* polynomial of the form

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

has a factorisation of the form (4.2), although the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ need not be distinct and may include non-real *complex numbers*. (Complex numbers are introduced in Unit I3.) It follows that a polynomial of degree n has at most n distinct roots.

For example,

$$x^4 - 6x^3 + 9x^2 + 4x - 12 = (x-2)(x-2)(x-3)(x+1).$$

Two useful consequences of this factorisation of $p(x)$ are

$$a_{n-1} = -(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \qquad (4.3)$$

and

$$a_0 = (-1)^n\alpha_1\alpha_2\cdots\alpha_n. \qquad (4.4)$$

In the above polynomial, the coefficient of x^3 is

$$-6 = -(2+2+3-1),$$

and the constant term is

$$-12 = (-1)^4 \times 2 \times 2 \times 3 \times (-1).$$

We obtain the first of these equations by comparing the coefficients of x^{n-1} on the two sides of the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n). \qquad (4.5)$$

When the expression on the right-hand side of equation (4.5) is multiplied out, each term in x^{n-1} arises by choosing the variable x from $n-1$ of the brackets, and the constant term from the remaining bracket. Choosing the constant term from the first bracket gives $-\alpha_1x^{n-1}$, choosing the constant term from the second bracket gives $-\alpha_2x^{n-1}$, and so on. Adding all these terms and comparing the resulting total coefficient with the coefficient of x^{n-1} on the left-hand side of equation (4.5) gives equation (4.3). Equation (4.4) is obtained by comparing the coefficients of x_0 on each side of equation (4.5).

Equations (4.3) and (4.4) relate the sum and product of the roots of the polynomial $p(x)$ to two of its coefficients, and they provide a useful check on the values of the roots found. Equation (4.4) is useful if you suspect that a polynomial of the form $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has n roots all of which are are integers; if they are, then each of the roots must be a factor of the constant coefficient a_0.

Example 4.4 Solve the following equation, given that all the solutions are integers.

$$p(x) = x^3 - 6x^2 - 9x + 14 = 0.$$

Solution Since all the roots of $p(x)$ are integers, the only possible roots are the factors of 14, that is, ±1, ±2, ±7, ±14. Considering these in turn, we obtain the following table.

x	1	−1	2	−2	7	−7	14	−14
$p(x)$	0	16	−20	0	0	−560	1456	−3780

For a cubic equation, once we have found three roots, we do not need to complete the rest of the table, so we could have omitted the last three columns here.

The only solutions are $x = 1$, $x = -2$ and $x = 7$. So,

$$x^3 - 6x^2 - 9x + 14 = (x-1)(x+2)(x-7). \quad ■$$

Exercise 4.6

(a) Solve each of the following equations, given that all their solutions are integers.

(i) $p(x) = x^3 - 3x^2 + 4 = 0$

(ii) $p(x) = x^3 - 9x^2 + 23x - 15 = 0$

(b) Determine a polynomial equation whose solutions are $1, 2, 3, -3$.

Further exercises

Exercise 4.7 Determine the expansions of the following expressions.

(a) $(a+3b)^4$

(b) $(1-t)^7$

Exercise 4.8 Find the coefficients of the following.

(a) a^3b^7 in the expansion of $(a+b)^{10}$

(b) x^{13} in the expansion of $(2+x)^{15}$

Exercise 4.9 Find the sum of each of the following finite geometric series.

(a) $3-1+\frac{1}{3}-\frac{1}{9}+\cdots-\frac{1}{3^{10}}$

(b) $1+\frac{a}{b}+\frac{a^2}{b^2}+\cdots+\frac{a^n}{b^n}$, where $a,b\in\mathbb{R}$, $a\neq b$ and $b\neq 0$

Exercise 4.10

(a) Show that $2x^3+x^2-13x+6$ has a factor $x-2$, and hence factorise this polynomial.

(b) Solve the equation $x^3+6x^2+3x-10=0$.

(c) Find the solutions for x in terms of y of the equation

$$x^2+x=y^2+y.$$

Exercise 4.11

(a) Find a cubic polynomial for which the sum of the roots is 0, the product of the roots is -30, and one root is 3.

(b) Find the other two roots of the polynomial found in part (a).